# Semantical conditions for the definability of functions and relations

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June 25, 2015

**Abstract**

Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages, let $R \in \mathcal{L}' - \mathcal{L}$ be a relation symbol, and let $\mathcal{K}$ be a class of $\mathcal{L}'$-structures. In this paper we present semantical conditions equivalent to the existence of an $\mathcal{L}$-formula $\varphi(\vec{x})$ such that $\mathcal{K} \vDash \varphi(\vec{x}) \leftrightarrow R(\vec{x})$, and $\varphi$ has a specific syntactical form (e.g., quantifier free, positive and quantifier free, existential horn, etc.). For each of these definability results for relations we also present an analogous version for the definability of functions. Several applications to natural definability questions in universal algebra have been included; most notably definability of principal congruences. The paper concludes with a look at term-interpolation in classes of structures with the same techniques used for definability. Here we obtain generalizations of two classical term-interpolation results: Pixley's theorem [14] for quasiprimal algebras, and the Baker-Pixley Theorem [2] for finite algebras with a majority term.

## Introduction

Let $\mathcal{L}$ be a first order language and $\mathcal{K}$ a class of $\mathcal{L}$-structures. If $R \in \mathcal{L}$ is an $n$-ary relation symbol, we say that a formula $\varphi(\vec{x})$ *defines* $R$ *in* $\mathcal{K}$ if

$$\mathcal{K} \vDash \varphi(\vec{x}) \leftrightarrow R(\vec{x}).$$

Let $f_1, \ldots, f_m \in \mathcal{L}$ be $n$-ary function symbols. Given an $\mathcal{L}$-structure $\mathbf{A}$, let $\vec{f}^{\mathbf{A}} : A^n \to A^m$ be the function defined by

$$\vec{f}^{\mathbf{A}}(\vec{a}) = (f_1^{\mathbf{A}}(\vec{a}), \ldots, f_m^{\mathbf{A}}(\vec{a})).$$

We say that a formula $\varphi(\vec{x}, \vec{z})$ *defines* $\vec{f}$ *in* $\mathcal{K}$ if

$$\mathcal{K} \vDash \varphi(\vec{x}, \vec{z}) \leftrightarrow \vec{f}(\vec{x}) = \vec{z}.$$

Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages, let $R \in \mathcal{L}' - \mathcal{L}$ be an $n$-ary relation symbol (resp. $f_1, \ldots, f_m \in \mathcal{L}' - \mathcal{L}$ be $n$-ary function symbols), and let $\mathcal{K}$ be a class of $\mathcal{L}'$-structures. Let $S$ be any of the following sets:

- {finite conjunctions of atomic $\mathcal{L}$-formulas},
- {positive open $\mathcal{L}$-formulas},
- {open Horn $\mathcal{L}$-formulas},
- {open $\mathcal{L}$-formulas},
- {primitive positive $\mathcal{L}$-formulas},
- {existential positive $\mathcal{L}$-formulas},
- {existential Horn $\mathcal{L}$-formulas},
- {existential $\mathcal{L}$-formulas}.

In this paper we give semantical conditions characterizing when $R$ (resp. $\vec{f}$) is definable in $\mathcal{K}$ by a formula of $S$. The results obtained provide a natural and unified way to handle familiar questions on definability of functions and relations in classes of structures. Being able to look at a great variety of definability questions within the same framework allows for a deeper understanding of the definability phenomena in general. Evidence of this is our finding of several new results on definability of principal congruences, and the generalizations of the Baker-Pixley Theorem [2] and of the Pixley Theorem [14]. The applications throughout the paper provide a good sample of how the results are put to work, in some cases providing direct proofs of known facts and in others discovering new theorems.

In Section 2 we study the definability of functions and relations by (positive) open formulas. We give some immediate applications to definability of relative principal congruences in quasivarieties (Proposition 5 and Corollary 6). In Section 3 we study the definability by open Horn formulas. In Section 4 we focus on the definability by conjunctions of atomic formulas. We give some consequences on definability of principal congruences and the Fraser-Horn property (Proposition 15 and Corollary 16). We also apply the characterizations to give new natural proofs of two results on the translation of (positive) open formulas to conjunctions of equations (Proposition 17 and Corollary 18). In Section 5 we address definability by existential formulas. Subsection 5.1 is concerned with primitive positive functions. As an application we characterize primitive positive functions in Stone algebras and De Morgan algebras. Section 6 is devoted to term interpolation of functions in classes. First we apply the previous results to characterize when a function is term valued by cases in a class (Theorems 28 and 30). We use this to give generalizations of Pixley's theorem [14] characterizing quasiprimal algebras as those finite algebras in which every function preserving the inner isomorphisms is a term function (Theorems 31 and 32). We conclude the section giving two generalizations of the Baker-Pixley Theorem [2] on the existence of terms representing functions in finite algebras with a majority term (Theorems 34 and 36).

Even though most results in the paper are true in more general contexts (via the same ideas), we have preferred to write the results in a more concise manner.

The intention is to provide the non-specialist a more accessible presentation, with the hope that he/she can find further natural applications in universal algebra.

# 1 Notation

As usual, $\mathbb{I}(\mathcal{K})$, $\mathbb{S}(\mathcal{K})$, $\mathbb{P}(\mathcal{K})$ and $\mathbb{P}_u(\mathcal{K})$ denote the classes of isomorphic images, substructures, direct products and ultraproducts of elements of $\mathcal{K}$. We write $\mathbb{P}_{\text{fin}}(\mathcal{K})$ to denote the class $\{\mathbf{A}_1 \times \ldots \times \mathbf{A}_n : n \geq 1$ and each $\mathbf{A}_i \in \mathcal{K}\}$. For a class of algebras $\mathcal{K}$ let $\mathbb{Q}(\mathcal{K})$ (resp. $\mathbb{V}(\mathcal{K})$) denote the *quasivariety* (resp. *variety*) *generated by* $\mathcal{K}$. If $\mathcal{L} \subseteq \mathcal{L}'$ are first order languages, for an $\mathcal{L}'$-structure $\mathbf{A}$ we use $\mathbf{A}_{\mathcal{L}}$ to denote the reduct of $\mathbf{A}$ to the language $\mathcal{L}$. If $\mathbf{A}, \mathbf{B}$ are $\mathcal{L}$-structures, we write $\mathbf{A} \leq \mathbf{B}$ to express that $\mathbf{A}$ is a substructure of $\mathbf{B}$.

Let $S_1, \ldots, S_k$ be non-empty sets, let $n \in N$. For $i = 1, \ldots, k$, let $f_i : S_i^n \to S_i$ be an $n$-ary operation on $S_i$. We use $f_1 \times \ldots \times f_k$ to denote the function $f_1 \times \ldots \times f_k : (S_1 \times \ldots \times S_k)^n \to S_1 \times \ldots \times S_k$ given by

$$f_1 \times \ldots \times f_k((a_1^1, \ldots, a_k^1), \ldots, (a_1^n, \ldots, a_k^n)) = (f_1(a_1^1, \ldots, a_1^n), \ldots, f_k(a_k^1, \ldots, a_k^n)).$$

Also, if $R_i \subseteq S_i^n$ are $n$-ary relations on $S_i$, then we write $R_1 \times \ldots \times R_k$ to denote the $n$-ary relation given by

$$R_1 \times \ldots \times R_k = \{(a_1^1, \ldots, a_k^1), \ldots, (a_1^n, \ldots, a_k^n)) : (a_i^1, \ldots, a_i^n) \in R_i,\ \ i = 1, \ldots, k\}.$$

We observe that if $\mathbf{S}_1, \ldots, \mathbf{S}_k$ are $\mathcal{L}$-structures and $f \in \mathcal{L}$ is an $n$-ary operation symbol, then $f^{\mathbf{S}_1} \times \ldots \times f^{\mathbf{S}_k} = f^{\mathbf{S}_1 \times \ldots \times \mathbf{S}_k}$. Also, if $R \in \mathcal{L}$ is an $n$-ary relation symbol, then $R^{\mathbf{S}_1} \times \ldots \times R^{\mathbf{S}_k} = R^{\mathbf{S}_1 \times \ldots \times \mathbf{S}_k}$.

For a quasivariety $\mathcal{Q}$ and $\mathbf{A} \in \mathcal{Q}$, we use $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$ to denote the lattice of relative congruences of $\mathbf{A}$. If $a, b \in A$, with $\mathbf{A} \in \mathcal{Q}$, let $\theta_{\mathcal{Q}}^{\mathbf{A}}(a, b)$ denote the *relative principal congruence* generated by $(a, b)$. When $\mathcal{Q}$ is a variety we drop the subscript and just write $\theta^{\mathbf{A}}(a, b)$. The quasivariety $\mathcal{Q}$ has *definable relative principal congruences* if there exists a first order formula $\varphi(x, y, z, w)$ in the language of $\mathcal{Q}$ such that

$$\theta_{\mathcal{Q}}^{\mathbf{A}}(a, b) = \{(c, d) : \mathcal{Q} \vDash \varphi(a, b, c, d)\}$$

for any $a, b \in A$, $\mathbf{A} \in \mathcal{Q}$. The quasivariety $\mathcal{Q}$ has the *relative congruence extension property* if for every $\mathbf{A} \leq \mathbf{B} \in \mathcal{Q}$ and $\theta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$ there is a $\delta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{B})$ such that $\theta = \delta \cap A^2$.

Let $\mathcal{Q}_{RFSI}$ (resp. $\mathcal{Q}_{RS}$) denote the class of relative finitely subdirectly irreducible (resp. simple) members of $\mathcal{Q}$. When $\mathcal{Q}$ is a variety, we write $\mathcal{Q}_{FSI}$ in place of $\mathcal{Q}_{RFSI}$, $\mathrm{Con}(\mathbf{A})$ in place of $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$, etc.

Let

$$\begin{aligned}
&\mathrm{At}(\mathcal{L}) = \{\text{atomic } \mathcal{L}\text{-formulas}\},\\
&\pm\mathrm{At}(\mathcal{L}) = \mathrm{At}(\mathcal{L}) \cup \{\neg\alpha : \alpha \in \mathrm{At}(\mathcal{L})\},\\
&\mathrm{Op}(\mathcal{L}) = \{\varphi : \varphi \text{ is an open } \mathcal{L}\text{-formula}\},\\
&\mathrm{OpHorn}(\mathcal{L}) = \{\varphi : \varphi \text{ is an open Horn } \mathcal{L}\text{-formula}\}.
\end{aligned}$$

If $S$ is a set of formulas, we define

$$
\begin{aligned}
&[\bigwedge S] = \{\varphi_1 \wedge \ldots \wedge \varphi_n : \varphi_1, \ldots, \varphi_n \in S,\ n \geq 1\}, \\
&[\bigvee S] = \{\varphi_1 \vee \ldots \vee \varphi_n : \varphi_1, \ldots, \varphi_n \in S,\ n \geq 1\}, \\
&[\forall S] = \{\forall x_1 \ldots \forall x_n \varphi : \varphi \in S,\ n \geq 0\}, \\
&[\exists S] = \{\exists x_1 \ldots \exists x_n \varphi : \varphi \in S,\ n \geq 0\}.
\end{aligned}
$$

# 2 Definability by (positive) open formulas

**Theorem 1** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $R \in \mathcal{L}' - \mathcal{L}$ be an $n$-ary relation symbol. For a class $\mathcal{K}$ of $\mathcal{L}'$-structures, the following are equivalent:*

*(1) There is a formula in $\mathrm{Op}(\mathcal{L})$ (resp. $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$) which defines $R$ in $\mathcal{K}$.*

*(2) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all isomorphisms (resp. homomorphisms) $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, and all $a_1, \ldots, a_n \in A_0$, we have that $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$ implies $(\sigma(a_1), \ldots, \sigma(a_n)) \in R^{\mathbf{B}}$.*

*Moreover, if $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $n$-generated substructures and each one is finite, then we can remove the operator $\mathbb{P}_u$ from (2).*

**Proof.** (1)⇒(2). Suppose that $\varphi(\vec{x}) \in \mathrm{Op}(\mathcal{L})$ (resp. $\varphi(\vec{x}) \in [\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$) defines $R$ in $\mathcal{K}$. Note that $\varphi(\vec{x})$ defines $R$ in $\mathbb{P}_u(\mathcal{K})$. Suppose that $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$, $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$ is an isomorphism (resp. a homomorphism), and fix $a_1, \ldots, a_n \in A_0$ such that $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$. Since

$$\mathbf{A} \vDash \varphi(\vec{a})$$

and $\varphi(\vec{x}) \in \mathrm{Op}(\mathcal{L})$, we have that

$$\mathbf{A}_0 \vDash \varphi(\vec{a}).$$

Since $\sigma$ is an isomorphism (resp. $\sigma$ is a homomorphism and $\varphi(\vec{x}) \in [\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$), we have that

$$\mathbf{B}_0 \vDash \varphi(\sigma(a_1), \ldots, \sigma(a_n)).$$

As $\varphi(\vec{x}) \in \mathrm{Op}(\mathcal{L})$, it follows that

$$\mathbf{B} \vDash \varphi(\sigma(a_1), \ldots, \sigma(a_n)),$$

and thus $(\sigma(a_1), \ldots, \sigma(a_n)) \in R^{\mathbf{B}}$.

(2)⇒(1). Let $\mathbf{A} \in \mathbb{P}_u(\mathcal{K})$ and $\vec{a} \in R^{\mathbf{A}}$. Define

$$\Delta^{\vec{a}, \mathbf{A}} = \{\alpha(\vec{x}) : \alpha \in \pm\mathrm{At}(\mathcal{L}),\ \mathbf{A} \vDash \alpha(\vec{a})\}$$

(resp. $\Delta^{\vec{a}, \mathbf{A}} = \{\alpha(\vec{x}) : \alpha \in \mathrm{At}(\mathcal{L})$ and $\mathbf{A} \vDash \alpha(\vec{a})\}$). Take $\mathbf{B} \in \mathbb{P}_u(\mathcal{K})$ and $\vec{b} \in B^n$ such that $\mathbf{B} \vDash \Delta^{\vec{a}, \mathbf{A}}(\vec{b})$. Let

$$\mathbf{A}_0 = \text{the substructure of } \mathbf{A}_{\mathcal{L}} \text{ generated by } a_1, \ldots, a_n,$$

$$\mathbf{B}_0 = \text{the substructure of } \mathbf{B}_{\mathcal{L}} \text{ generated by } b_1, \ldots, b_n.$$

Since $\mathbf{B} \vDash \Delta^{\vec{a},\mathbf{A}}(\vec{b})$, we have that

$$a_1 \mapsto b_1, \ldots, a_n \mapsto b_n, \quad f_1^{\mathbf{A}}(\vec{a}) \mapsto c_1, \ldots, f_m^{\mathbf{A}}(\vec{a}) \mapsto c_m$$

extends to an isomorphism (resp. homomorphism) from $\mathbf{A}_0$ onto $\mathbf{B}_0$, which by (2) says that $\vec{b} \in R^{\mathbf{B}}$. So we have proved that

$$\mathbb{P}_u(\mathcal{K}) \vDash \left( \bigwedge_{\alpha \in \Delta^{\vec{a},\mathbf{A}}} \alpha(\vec{x}) \right) \to R\left(\vec{x}\right).$$

By compactness, there is a finite subset $\Delta_0^{\vec{a},\mathbf{A}} \subseteq \Delta^{\vec{a},\mathbf{A}}$ such that

$$\mathbb{P}_u(\mathcal{K}) \vDash \left( \bigwedge_{\alpha \in \Delta_0^{\vec{a},\mathbf{A}}} \alpha(\vec{x}) \right) \to R(\vec{x}).$$

Next note that

$$\mathbb{P}_u(\mathcal{K}) \vDash \left( \bigvee_{\mathbf{A} \in \mathbb{P}_u(\mathcal{K}),\ \vec{a} \in R^{\mathbf{A}}} \ \bigwedge_{\alpha \in \Delta_0^{\vec{a},\mathbf{A}}} \alpha(\vec{x}) \right) \leftrightarrow R(\vec{x}),$$

which by compactness says that

$$\mathbb{P}_u(\mathcal{K}) \vDash \left( \bigvee_{i=1}^{k} \ \bigwedge_{\alpha \in \Delta_0^{\vec{a}_i,\mathbf{A}_i}} \alpha(\vec{x}) \right) \leftrightarrow R(\vec{x})$$

for some $\mathbf{A}_1, \ldots, \mathbf{A}_k \in \mathbb{P}_u(\mathcal{K})$, $\vec{a}_1 \in R^{\mathbf{A}_1}, \ldots, \vec{a}_k \in R^{\mathbf{A}_k}$.

Now we prove the moreover part. Suppose $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $n$-generated substructures and each one is finite. Thus, there is a finite list of atomic $\mathcal{L}$-formulas $\alpha_1(\vec{x}), \ldots, \alpha_k(\vec{x})$ such that for every atomic $\mathcal{L}$-formula $\alpha(\vec{x})$, there is $j \in \{1, \ldots, k\}$ satisfying $\mathcal{K} \vDash \alpha(\vec{x}) \leftrightarrow \alpha_j(\vec{x})$. Assume (2) holds without the ultraproduct operator, we prove (1). By considerations similar to the above we have that

$$\mathcal{K} \vDash \left( \bigvee_{\mathbf{A} \in \mathcal{K},\ \vec{a} \in R^{\mathbf{A}}} \ \bigwedge_{\alpha \in \Delta^{\vec{a},\mathbf{A}}} \alpha(\vec{x}) \right) \leftrightarrow R(\vec{x}).$$

Since each $\Delta^{\vec{a},\mathbf{A}}$ can be supposed to be included in

$$\{\alpha_1(\vec{x}), \ldots, \alpha_k(\vec{x})\} \cup \{\neg\alpha_1(\vec{x}), \ldots, \neg\alpha_k(\vec{x})\}$$

(resp. $\{\alpha_1(\vec{x}), \ldots, \alpha_k(\vec{x})\}$), we have (after removing redundancies) that,

$$\bigvee_{\mathbf{A} \in \mathcal{K},\ \vec{a} \in R^{\mathbf{A}}} \ \bigwedge_{\alpha \in \Delta^{\vec{a},\mathbf{A}}} \alpha(\vec{x})$$

is a first order formula.

Finally, note that the remaining implication is already taken care of by (1)⇒(2) above. ■

Here is a direct consequence of the above theorem.

**Corollary 2** *Let $\mathcal{K}$ be any class of $\mathcal{L}$-algebras contained in a locally finite variety. Suppose $\mathbf{A} \to R^{\mathbf{A}}$ is a map which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary relation $R^{\mathbf{A}} \subseteq A^n$. The following are equivalent:*

*(1) There is a formula in $\mathrm{Op}(\mathcal{L})$ (resp. $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$) which defines $R$ in $\mathcal{K}$.*

*(2) For all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all isomorphisms (resp. homomorphisms) $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, and all $a_1, \ldots, a_n \in A_0$ we have that $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$ implies $(\sigma(a_1), \ldots, \sigma(a_n)) \in R^{\mathbf{B}}$.*

**Proof.** Apply the moreover part of Theorem 1 to the class $\{(\mathbf{A}, R^{\mathbf{A}}) : \mathbf{A} \in \mathcal{K}\}$. ■

As we shall see next it is easy to derive the functional version of Theorem 1.

**Theorem 3** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $f_1, \ldots, f_m \in \mathcal{L}' - \mathcal{L}$ be $n$-ary function symbols. For a class $\mathcal{K}$ of $\mathcal{L}'$-structures, the following are equivalent:*

*(1) There is a formula in $\mathrm{Op}(\mathcal{L})$ (resp. $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$) which defines $\vec{f}$ in $\mathcal{K}$.*

*(2) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all isomorphisms (resp. homomorphisms) $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, and all $a_1, \ldots, a_n \in A_0$ such that $f_1^{\mathbf{A}}(\vec{a}), \ldots, f_m^{\mathbf{A}}(\vec{a}) \in A_0$, we have $\sigma(f_i^{\mathbf{A}}(\vec{a})) = f_i^{\mathbf{B}}(\sigma(a_1), \ldots, \sigma(a_n))$ for all $i \in \{1, \ldots, m\}$.*

*Moreover, if $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $(n+m)$-generated substructures and each one is finite, then we can remove the operator $\mathbb{P}_u$ from (2).*

**Proof.** The implication (1)⇒(2) is analogous to (1)⇒(2) of Theorem 1.

We prove (2)⇒(1). Let $R \notin \mathcal{L}'$ be an $(n+m)$-ary relational symbol. For each $\mathbf{A} \in \mathcal{K}$ let

$$R^{\mathbf{A}} = \{\big(a_1, \ldots, a_n, f_1^{\mathbf{A}}(\vec{a}), \ldots, f_m^{\mathbf{A}}(\vec{a})\big) : a_1, \ldots, a_n \in A\}.$$

Define the following class of $\mathcal{L}' \cup \{R\}$-structures

$$\mathcal{K}_R = \{\big(\mathbf{A}, R^{\mathbf{A}}\big) : \mathbf{A} \in \mathcal{K}\}.$$

It is not hard to check that the isomorphism (resp. homomorphism) version of (2) in Theorem 1 holds for the languages $\mathcal{L} \subseteq \mathcal{L}' \cup \{R\}$, the relation symbol $R$ and the class $\mathcal{K}_R$. Thus there is $\varphi(x_1, \ldots, x_n, z_1, \ldots, z_m) \in \mathrm{Op}(\mathcal{L})$ (resp. $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$) such that $\mathcal{K}_R \vDash \varphi(\vec{x}, \vec{z}) \leftrightarrow R(\vec{x}, \vec{z})$. Now as $\mathcal{K}_R \vDash \vec{f}(\vec{x}) = \vec{z} \leftrightarrow R(\vec{x}, \vec{z})$, it immediately follows that $\varphi$ defines $\vec{f}$ in $\mathcal{K}_R$. Hence $\varphi$ defines $\vec{f}$ in $\mathcal{K}$.

Next we prove the moreover part. Suppose $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $(n+m)$-generated substructures and each one is finite. Assume (2) without $\mathbb{P}_u$. Let $\mathcal{K}_R$ be as in the first part of the proof. Note that $(\mathcal{K}_R)_{\mathcal{L}} = \mathcal{K}_{\mathcal{L}}$ and thus we can apply the moreover part of Theorem 1 to obtain $\varphi \in \mathrm{Op}(\mathcal{L})$ (resp. $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$) defining $R$ in $\mathcal{K}_R$. Clearly $\varphi$ defines $\vec{f}$ in $\mathcal{K}$. The remaining implication is immediate by (1)⇒(2). ■

**Corollary 4** *Let $\mathcal{K}$ be any class of $\mathcal{L}$-algebras contained in a locally finite variety. Suppose $\mathbf{A} \to f^{\mathbf{A}}$ is a map which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary operation $f^{\mathbf{A}} : A^n \to A$. The following are equivalent:*

*(1) There is a formula in $\mathrm{Op}(\mathcal{L})$ (resp. $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$) which defines $f$ in $\mathcal{K}$.*

*(2) For all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all isomorphisms (resp. homomorphisms) $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, and all $a_1, \ldots, a_n \in A_0$ such that $f^{\mathbf{A}}(\vec{a}) \in A_0$, we have $\sigma(f^{\mathbf{A}}(\vec{a})) = f^{\mathbf{B}}(\sigma(a_1), \ldots, \sigma(a_n))$.*

**Proof.** Apply the moreover part of Theorem 3 to the class $\{(\mathbf{A}, f^{\mathbf{A}}) : \mathbf{A} \in \mathcal{K}\}$. ■

Let $\mathbf{3} = (\{0, 1/2, 1\}, \max, \min, ^*, 0, 1)$, where $0^* = 1$ and $(1/2)^* = 1^* = 0$. Of course, $\mathbf{3}$ is the three-element Stone algebra (see [1]). Note that the only non trivial homomorphism between subalgebras of $\mathbf{3}$ is the map $^{**} : \{0, 1/2, 1\} \to \{0, 1\}$. Thus the above corollary applied to the class $\mathcal{K} = \{\mathbf{3}\}$ says that a function $f : \{0, 1/2, 1\}^n \to \{0, 1/2, 1\}$ is definable in $\mathbf{3}$ by a positive open formula in the language of $\mathbf{3}$ iff $f(x_1, \ldots, x_n)^{**} = f(x_1^{**}, \ldots, x_n^{**})$, for any $x_1, \ldots, x_n \in \{0, 1/2, 1\}$.

## Applications to definable principal congruences

We apply the above results to give natural proofs of two results on definability of relative principal congruences in quasivarieties.

**Proposition 5** *Let $\mathcal{Q}$ be a quasivariety with definable relative principal congruences and let $\mathcal{L}$ be the language of $\mathcal{Q}$. The following are equivalent:*

*(1) There is a formula in $[\forall \mathrm{Op}(\mathcal{L})]$ defining relative principal congruences in $\mathcal{Q}$.*

*(2) There is a formula in $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$ defining relative principal congruences in $\mathcal{Q}$.*

*(3) $\mathcal{Q}$ has the relative congruence extension property.*

**Proof.** (1)⇒(3). We need the following fact proved in [4].

- A quasivariety $\mathcal{Q}$ has the relative congruence extension property if for every $\mathbf{A}, \mathbf{B} \in \mathcal{Q}$ with $\mathbf{A} \leq \mathbf{B}$ and for all $a, b \in A$ we have that $\theta_{\mathcal{Q}}^{\mathbf{A}}(a, b) = \theta_{\mathcal{Q}}^{\mathbf{B}}(a, b) \cap A^2$.

Note that it is always the case that $\theta_{\mathcal{Q}}^{\mathbf{A}}(a,b) \subseteq \theta_{\mathcal{Q}}^{\mathbf{B}}(a,b) \cap A^2$. So, as formulas in $[\forall \mathrm{Op}(\mathcal{L})]$ are preserved by subalgebras, (1) implies that $\theta_{\mathcal{Q}}^{\mathbf{A}}(a,b) = \theta_{\mathcal{Q}}^{\mathbf{B}}(a,b) \cap A^2$ for every $\mathbf{A} \leq \mathbf{B} \in \mathcal{Q}$. Thus the fact cited above yields (3).

(3)$\Rightarrow$(2). We use the following well known fact.

(i) For all $\mathbf{A}, \mathbf{B} \in \mathcal{Q}$ and all homomorphisms $\sigma : \mathbf{A} \to \mathbf{B}$ we have that $(a,b) \in \theta_{\mathcal{Q}}^{\mathbf{A}}(c,d)$ implies $(\sigma(a), \sigma(b)) \in \theta_{\mathcal{Q}}^{\mathbf{B}}(\sigma(c), \sigma(d))$.

For each $\mathbf{A} \in \mathcal{Q}$, let

$$R^{\mathbf{A}} = \{(a,b,c,d) : (a,b) \in \theta_{\mathcal{Q}}^{\mathbf{A}}(c,d)\},$$

and define

$$\mathcal{K} = \{(\mathbf{A}, R^{\mathbf{A}}) : \mathbf{A} \in \mathcal{Q}\}.$$

Since $\mathcal{Q}$ has definable relative principal congruences, $\mathcal{K}$ is a first order class and hence $\mathbb{P}_u(\mathcal{K}) \subseteq \mathcal{K}$. Thus, in order to prove that (2) of Theorem 1 holds we need to check that:

(ii) For all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all homomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$ and all $a,b,c,d \in A_0$, we have that $(a,b,c,d) \in R^{\mathbf{A}}$ implies $(\sigma(a), \sigma(b), \sigma(c), \sigma(d)) \in R^{\mathbf{B}}$.

Or, equivalently:

(iii) For all $\mathbf{A}, \mathbf{B} \in \mathcal{Q}$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all homomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$ and all $a,b,c,d \in A_0$, we have that $(a,b) \in \theta_{\mathcal{Q}}^{\mathbf{A}}(c,d)$ implies $(\sigma(a), \sigma(b)) \in \theta_{\mathcal{Q}}^{\mathbf{B}}(\sigma(c), \sigma(d))$.

Since $\mathcal{V}$ has the congruence extension property, we can replace in (iii) the occurrence of "$(a,b) \in \theta_{\mathcal{Q}}^{\mathbf{A}}(c,d)$" by "$(a,b) \in \theta_{\mathcal{Q}}^{\mathbf{A}_0}(c,d)$". Hence (iii) follows from (i).

(2)$\Rightarrow$(1). This is trivial. ■

**Corollary 6** *Let $\mathcal{Q}$ be a locally finite quasivariety with the relative congruence extension property. Let $\mathcal{L}$ be the language of $\mathcal{Q}$. Then there is a formula in $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$ which defines the relative principal congruences in $\mathcal{Q}$.*

**Proof.** The proof is similar to (3)$\Rightarrow$(2) in the above proof, but applying Corollary 2 in place of Theorem 1. ■

The above corollary is proved in [3] for the case in which $\mathcal{Q}$ is a finitely generated variety.

# 3 Definability by open Horn formulas

**Theorem 7** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $R \in \mathcal{L}' - \mathcal{L}$ be an $n$-ary relation symbol. Let $\mathcal{K}$ be any class of $\mathcal{L}'$-structures. The following are equivalent:*

*(1) There is a formula in* OpHorn($\mathcal{L}$) *which defines* $R$ *in* $\mathcal{K}$.

*(2) For all* $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u\mathbb{P}_{\text{fin}}(\mathcal{K})$*, all* $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$*,* $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$*, all isomorphisms* $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$*, and all* $a_1, \ldots, a_n \in A_0$*, we have that* $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$ *implies* $(\sigma(a_1), \ldots, \sigma(a_n)) \in R^{\mathbf{B}}$.

*Moreover, if* $\mathcal{K}_{\mathcal{L}}$ *has finitely many isomorphism types of* $n$*-generated substructures and each one is finite, then we can remove the operator* $\mathbb{P}_u$ *from (2).*

**Proof.** (1)⇒(2). Note that if $\varphi(\vec{x}) \in \text{OpHorn}(\mathcal{L})$ defines $R$ in $\mathcal{K}$, then $\varphi$ defines $R$ in $\mathbb{P}_u\mathbb{P}_{\text{fin}}(\mathcal{K})$ as well. Now we can repeat the argument of (1)⇒(2) in the proof of Theorem 1.

(2)⇒(1). Applying Theorem 1 to the class $\mathbb{P}_{\text{fin}}(\mathcal{K})$ we have that there is an open $\mathcal{L}$-formula $\varphi$ which defines $R$ in $\mathbb{P}_{\text{fin}}(\mathcal{K})$. W.l.o.g. we can suppose that

$$\varphi = \bigwedge_{j=1}^{r} \left( \pi_j \to \bigvee_{i=1}^{k_j} \alpha_i^j \right) \wedge \bigwedge_{j=1}^{l} \bigvee_{i=1}^{u_j} \neg\beta_i^j$$

with $r, k_j, u_j \geq 1$, $l \geq 0$, each $\pi_j$ in $[\bigwedge \text{At}(\mathcal{L})]$ and the formulas $\alpha_i^j, \beta_i^j$ in $\text{At}(\mathcal{L})$. Note that for $\mathbf{A} \in \mathbb{P}_{\text{fin}}(\mathcal{K})$ and $\vec{a} \in R^{\mathbf{A}}$ we have that

$$\mathbf{A} \vDash \left( \bigwedge_{j=1}^{l} \bigvee_{i=1}^{u_j} \neg\beta_i^j \right)(\vec{a}).$$

Let

$$S = \{(s_1, \ldots, s_r) : 1 \leq s_j \leq k_j, \quad j = 1, \ldots, r\}.$$

We claim that there is $s \in S$ such that $\bigwedge_{j=1}^{r} \left( \pi_j \to \alpha_{s_j}^j \right) \wedge \bigwedge_{j=1}^{l} \bigvee_{i=1}^{u_j} \neg\beta_i^j$ defines $R$ in $\mathcal{K}$. For the sake of contradiction assume that this is not the case. Then for each $s \in S$ there are $\mathbf{A}_s \in \mathcal{K}$, $\vec{a}_s = (a_{s1}, \ldots, a_{sn}) \in R^{\mathbf{A}_s}$ and $j_s$ such that

$$\mathbf{A}_s \vDash \neg \left( \pi_{j_s} \to \alpha_{s_{j_s}}^{j_s} \right)(\vec{a}_s),$$

or equivalently

$$\mathbf{A}_s \vDash \pi_{j_s}(\vec{a}_s) \wedge \neg\alpha_{s_{j_s}}^{j_s}(\vec{a}_s). \tag{i}$$

Let $p_1 = (a_{s1})_{s \in S}, \ldots, p_n = (a_{sn})_{s \in S}$. Note that $\vec{p} \in R^{\Pi_S \mathbf{A}_s}$, and as $\varphi$ defines $R$ in $\mathbb{P}_{\text{fin}}(\mathcal{K})$, we have

$$\prod_{s \in S} \mathbf{A}_s \vDash \varphi(\vec{p}).$$

This implies

$$\prod_{s \in S} \mathbf{A}_s \vDash \bigwedge_{j=1}^{r} \left( \pi_j \to \bigvee_{i=1}^{k_j} \alpha_i^j \right)(\vec{p}).$$

By this and (i) we have

$$\prod_{s\in S}\mathbf{A}_s \vDash \left(\bigwedge_{j=1}^{r}\pi_j\right)(\vec{p}).$$

Hence for each $j\in\{1,\ldots,r\}$ there is an $s_j$ such that

$$\prod_{s\in S}\mathbf{A}_s \vDash \alpha^j_{s_j}(\vec{p}).$$

Let $s=(s_1,\ldots,s_r)$. Then we have that

$$\mathbf{A}_s \vDash \alpha^{j_s}_{s_{j_s}}(\vec{a}_s),$$

which contradicts (i).

To prove the moreover part, suppose $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $n$-generated substructures and each one is finite. Assume (2) without the ultraproduct operator. We prove (1). Note that our hypothesis on $\mathcal{K}_{\mathcal{L}}$ implies that there are atomic $\mathcal{L}$-formulas $\alpha_1(x_1,\ldots,x_n),\ldots,\alpha_k(x_1,\ldots,x_n)$ such that for every atomic $\mathcal{L}$-formula $\alpha(\vec{x})$ there is $j$ satisfying $\mathcal{K}\vDash\alpha(\vec{x})\leftrightarrow\alpha_j(\vec{x})$. Since atomic formulas are preserved by direct products and by direct factors, we have that for every atomic $\mathcal{L}$-formula $\alpha(\vec{x})$, there is $j$ such that $\mathbb{P}_{\text{fin}}(\mathcal{K})\vDash\alpha(\vec{x})\leftrightarrow\alpha_j(\vec{x})$. This implies that $\mathbb{P}_{\text{fin}}(\mathcal{K})_{\mathcal{L}}$ has finitely many isomorphism types of $n$-generated substructures and each one is finite. By Theorem 1 there is an open $\mathcal{L}$-formula $\varphi$ which defines $R$ in $\mathbb{P}_{\text{fin}}(\mathcal{K})$. Now we can proceed as in the first part of this proof. ■

By a *trivial* $\mathcal{L}$-structure we mean a structure $\mathbf{A}$ such that $A=\{a\}$ and $(a,\ldots,a)\in R^{\mathbf{A}}$, for every $R\in\mathcal{L}$. Recall that a *strict Horn formula* is a Horn formula that has exactly one non-negated atomic formula in each of its clauses. Let us write $\mathrm{OpStHorn}(\mathcal{L})$ for the set of open strict Horn $\mathcal{L}$-formulas.

**Remark 8** *Theorem 7 holds if we replace in (1)* $\mathrm{OpHorn}(\mathcal{L})$ *by* $\mathrm{OpStHorn}(\mathcal{L})$ *and add the following requirement to (2):*

*For all $\mathbf{A}\in\mathbb{P}_u(\mathcal{K})_{\mathcal{L}}$ with a trivial substructure $\{a\}$, we have $(a,\ldots,a)\in R^{\mathbf{A}}$.*

**Proof.** (1)⇒(2). Observe that formulas in $\mathrm{OpStHorn}(\mathcal{L})$ are always satisfied in trivial structures.

To see (2)⇒(1) note that by Theorem 7 there is a formula

$$\varphi(\vec{x})=\bigwedge_{j=1}^{r}(\pi_j\to\alpha_j)\wedge\bigwedge_{j=1}^{l}\bigvee_{i=1}^{u_j}\neg\beta^j_i$$

(with $r,u_j\geq 1$, $l\geq 0$, each $\pi_j$ in $[\bigwedge\mathrm{At}(\mathcal{L})]$ and the formulas $\alpha_j,\beta^j_i$ in $\mathrm{At}(\mathcal{L})$) which defines $R$ in $\mathcal{K}$. Note that $\varphi(\vec{x})$ also defines $R$ in $\mathbb{P}_u(\mathcal{K})$. Assume $l\geq 1$ and

suppose $\mathbf{A} \in \mathbb{P}_u(\mathcal{K})_{\mathcal{L}}$ has a trivial substructure $\{a\}$. Note that the additional condition of (2) says that

$$\mathbf{A} \vDash \varphi(a, \ldots, a),$$

which is absurd since $l \geq 1$. Thus we have proved that there is no trivial substructure in $\mathbb{P}_u(\mathcal{K})_{\mathcal{L}}$. Hence

$$\mathbb{P}_u(\mathcal{K}) \vDash \bigvee_{\alpha(z_1) \in \mathrm{At}(\mathcal{L})} \neg\alpha(z_1),$$

which by compactness says that

$$\mathcal{K} \vDash \neg\tilde{\alpha}_1(z_1) \vee \ldots \vee \neg\tilde{\alpha}_k(z_1)$$

for some atomic $\mathcal{L}$-formulas $\tilde{\alpha}_1(z_1), \ldots, \tilde{\alpha}_k(z_1)$. Now it is easy to check that the formula

$$\bigwedge_{j=1}^{r} (\pi_j \to \alpha_j) \wedge \bigwedge_{j=1}^{l} \bigwedge_{t=1}^{k} \left( \bigwedge_{i=1}^{u_j} \beta_i^j \to \tilde{\alpha}_t \right)$$

defines $R$ in $\mathcal{K}$. ■

Here is the functional version of Theorem 7.

**Theorem 9** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $f_1, \ldots, f_m \in \mathcal{L}' - \mathcal{L}$ be $n$-ary function symbols. For a class $\mathcal{K}$ of $\mathcal{L}'$-structures, the following are equivalent:*

*(1) There is a formula in* OpHorn($\mathcal{L}$) *(resp.* OpStHorn($\mathcal{L}$)*) which defines $\vec{f}$ in $\mathcal{K}$.*

*(2) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u\mathbb{P}_{\mathrm{fin}}(\mathcal{K})$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all isomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, and all $a_1, \ldots, a_n \in A_0$ such that $f_1^{\mathbf{A}}(\vec{a}), \ldots, f_m^{\mathbf{A}}(\vec{a}) \in A_0$, we have that $\sigma(f_i^{\mathbf{A}}(\vec{a})) = f_i^{\mathbf{B}}(\sigma(a_1), \ldots, \sigma(a_n))$ for all $i \in \{1, \ldots, m\}$. (For all $\mathbf{A} \in \mathcal{K}_{\mathcal{L}}$, and every trivial substructure $\{a\}$, we have $f_i^{\mathbf{A}}(a, \ldots, a) = a$ for all $i \in \{1, \ldots, m\}$.)*

*Moreover, if $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $(n+m)$-generated substructures and each one is finite, then we can remove the operator $\mathbb{P}_u$ in (2).*

**Proof.** This can be proved applying Theorem 7 in the same way as we applied 1 to prove Theorem 3. ■

**Corollary 10** *Let $\mathcal{K}$ be any class of $\mathcal{L}$-algebras contained in a locally finite variety. Suppose $\mathbf{A} \to f^{\mathbf{A}}$ is a map which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary operation $f^{\mathbf{A}} : A^n \to A$. The following are equivalent:*

*(1) There is a conjunction of $\mathcal{L}$-formulas of the form $(\bigwedge_i p_i = q_i) \to r = s$ which defines $f$ in $\mathcal{K}$.*

*(2) The following conditions hold:*

(a) *For all* $\mathbf{A} \in \mathcal{K}$ *and all* $\{a\} \leq \mathbf{A}$*, we have* $f^{\mathbf{A}}(a, \ldots, a) = a$.

(b) *For all* $\mathbf{S}_1 \leq \mathbf{A}_1 \times \ldots \times \mathbf{A}_k$, $\mathbf{S}_2 \leq \mathbf{B}_1 \times \ldots \times \mathbf{B}_l$*, with* $\mathbf{A}_1, \ldots, \mathbf{A}_k, \mathbf{B}_1, \ldots, \mathbf{B}_l \in \mathcal{K}$*, all isomorphisms* $\sigma : \mathbf{S}_1 \to \mathbf{S}_2$*, and all* $p_1, \ldots, p_n \in S_1$ *such that* $f^{\mathbf{A}_1} \times \ldots \times f^{\mathbf{A}_k}(p_1, \ldots, p_n) \in S_1$*, we have* $\sigma(f^{\mathbf{A}_1} \times \ldots \times f^{\mathbf{A}_k}(p_1, \ldots, p_n)) = f^{\mathbf{B}_1} \times \ldots \times f^{\mathbf{B}_l}(\sigma(p_1), \ldots, \sigma(p_n))$.

**Proof.** Apply the moreover part of Theorem 9 to the class $\{(\mathbf{A}, f^{\mathbf{A}}) : \mathbf{A} \in \mathcal{K}\}$. ■

# 4 Definability by conjunctions of atomic formulas

**Theorem 11** *Let* $\mathcal{L} \subseteq \mathcal{L}'$ *be first order languages and let* $R \in \mathcal{L}' - \mathcal{L}$ *be an* $n$*-ary relation symbol. Let* $\mathcal{K}$ *be any class of* $\mathcal{L}'$*-structures The following are equivalent:*

(1) *There is a formula in* $[\bigwedge \mathrm{At}(\mathcal{L})]$ *which defines* $R$ *in* $\mathcal{K}$.

(2) *For all* $\mathbf{A} \in \mathbb{P}_u\mathbb{P}_{\mathrm{fin}}(\mathcal{K})$*, all* $\mathbf{B} \in \mathbb{P}_u(\mathcal{K})$, $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$*, all homomorphisms* $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$*, and all* $a_1, \ldots, a_n \in A_0$*, we have that* $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$ *implies* $(\sigma(a_1), \ldots, \sigma(a_n)) \in R^{\mathbf{B}}$.

*Moreover, if* $\mathcal{K}_{\mathcal{L}}$ *has finitely many isomorphism types of* $n$*-generated substructures and each one is finite, then we can remove the operator* $\mathbb{P}_u$ *from (2).*

**Proof.** (1)⇒(2). This is analogous to the proof of (1)⇒(2) in Theorem 7.

(2)⇒(1). Note that (2) holds also when $\mathbf{B}$ is in $\mathbb{ISPP}_u(\mathcal{K})$. Since $\mathbb{P}_u\mathbb{P}_{\mathrm{fin}}(\mathcal{K}) \subseteq \mathbb{ISPP}_u(\mathcal{K})$, applying Theorem 1 to the class $\mathbb{P}_{\mathrm{fin}}(\mathcal{K})$ we have that there is an $\mathcal{L}$-formula $\varphi = \pi_1 \vee \ldots \vee \pi_k$, with each $\pi_i$ a conjunction of atomic formulas, which defines $R$ in $\mathbb{P}_{\mathrm{fin}}(\mathcal{K})$. Using the same argument as in the proof of Theorem 7 we can prove that there is $j$ such that $\pi_j$ defines $R$ in $\mathcal{K}$.

The proof of the moreover part is similar to the corresponding part of the proof of Theorem 7. ■

**Corollary 12** *Let* $\mathcal{K}$ *be a class of* $\mathcal{L}$*-algebras contained in a locally finite variety. Suppose* $\mathbf{A} \to R^{\mathbf{A}}$ *is a map which assigns to each* $\mathbf{A} \in \mathcal{K}$ *an* $n$*-ary relation* $R^{\mathbf{A}} \subseteq A^n$*. The following are equivalent:*

(1) *There is a formula in* $[\bigwedge \mathrm{At}(\mathcal{L})]$ *which defines* $R$ *in* $\mathcal{K}$.

(2) *For all* $k \in \mathbb{N}$*, all* $\mathbf{A}_1, \ldots, \mathbf{A}_k, \mathbf{A} \in \mathcal{K}$*, all* $\mathbf{S} \leq \mathbf{A}_1 \times \ldots \times \mathbf{A}_k$*, all homomorphisms* $\sigma : \mathbf{S} \to \mathbf{A}$*, and all* $p_1, \ldots, p_n \in S$*, we have that* $(p_1, \ldots, p_n) \in R^{\mathbf{A}_1} \times \ldots \times R^{\mathbf{A}_k}$ *implies* $(\sigma(p_1), \ldots, \sigma(p_n)) \in R^{\mathbf{A}}$.

**Proof.** Apply the moreover part of Theorem 11 to the class $\{(\mathbf{A}, R^{\mathbf{A}}) : \mathbf{A} \in \mathcal{K}\}$. ■

Next are the results for definability of functions with conjunction of atomic formulas.

**Theorem 13** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $f_1, \ldots, f_m \in \mathcal{L}' - \mathcal{L}$ be $n$-ary function symbols. For a class $\mathcal{K}$ of $\mathcal{L}'$-structures, the following are equivalent*

*(1) There is a formula $\varphi$ in $[\bigwedge \mathrm{At}(\mathcal{L})]$ which defines $\vec{f}$ in $\mathcal{K}$.*

*(2) For all $\mathbf{A} \in \mathbb{P}_u\mathbb{P}_{\text{fin}}(\mathcal{K})$, all $\mathbf{B} \in \mathbb{P}_u(\mathcal{K})$, $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all homomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, and all $a_1, \ldots, a_n \in A_0$ such that $f_1^{\mathbf{A}}(\vec{a}), \ldots, f_m^{\mathbf{A}}(\vec{a}) \in A_0$, we have $\sigma(f_i^{\mathbf{A}}(\vec{a})) = f_i^{\mathbf{B}}(\sigma(a_1), \ldots, \sigma(a_n))$ for all $i \in \{1, \ldots, m\}$.*

*Moreover, if $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $(n+m)$-generated substructures and each one is finite, then we can remove the operator $\mathbb{P}_u$ from (2).*

**Proof.** This can be proved applying Theorem 11 in the same way as we applied 1 to prove Theorem 3. ■

**Corollary 14** *Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras contained in a locally finite variety. Suppose $\mathbf{A} \to f^{\mathbf{A}}$ is a map which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary operation $f^{\mathbf{A}} : A^n \to A$. The following are equivalent:*

*(1) There is a formula in $[\bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathcal{K}$.*

*(2) For all $k \in \mathbb{N}$, all $\mathbf{A}_1, \ldots, \mathbf{A}_k, \mathbf{A} \in \mathcal{K}$, all $\mathbf{S} \leq \mathbf{A}_1 \times \ldots \times \mathbf{A}_k$, all homomorphisms $\sigma : \mathbf{S} \to \mathbf{A}$, and all $p_1, \ldots, p_n \in S$ such that $f^{\mathbf{A}_1} \times \ldots \times f^{\mathbf{A}_k}(p_1, \ldots, p_n) \in S$, we have $\sigma(f^{\mathbf{A}_1} \times \ldots \times f^{\mathbf{A}_k}(p_1, \ldots, p_n)) = f^{\mathbf{A}}(\sigma(p_1), \ldots, \sigma(p_n))$.*

**Proof.** Apply the moreover part of Theorem 13 to the class $\{(\mathbf{A}, f^{\mathbf{A}}) : \mathbf{A} \in \mathcal{K}\}$. ■

## Applications to definable principal congruences

A variety $\mathcal{V}$ in a language $\mathcal{L}$ has *equationally definable principal congruences* if there exists a formula $\varphi(x, y, z, w) \in [\bigwedge \mathrm{At}(\mathcal{L})]$ such that

$$\theta^{\mathbf{A}}(a, b) = \{(c, d) : \mathcal{V} \vDash \varphi(a, b, c, d)\}$$

for any $a, b \in A$, $\mathbf{A} \in \mathcal{V}$. (This notion is called *equationally definable principal congruences in the restricted sense* in [5].) The variety $\mathcal{V}$ has the *Fraser-Horn property* if for every $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{V}$ and $\theta \in \mathrm{Con}(\mathbf{A}_1 \times \mathbf{A}_2)$, there are $\theta_1 \in \mathrm{Con}(\mathbf{A}_1)$ and $\theta_2 \in \mathrm{Con}(\mathbf{A}_2)$ such that $\theta = \theta_1 \times \theta_2$ (i.e., algebras in $\mathcal{V}$ do not have skew congruences).

Here is an interesting application of Theorem 11.

**Proposition 15** *Let $\mathcal{V}$ be a variety with definable principal congruences. The following are equivalent:*

*(1) $\mathcal{V}$ has equationally definable principal congruences.*

*(2) $\mathcal{V}$ has the congruence extension property and the Fraser-Horn property.*

**Proof.** It is well known (see [5]) that:

(i) A variety has the Fraser-Horn property iff for all $n \in \mathbb{N}$, all $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{V}$, and all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ we have

$$\theta^{\mathbf{A}_1 \times \ldots \times \mathbf{A}_n}((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \theta^{\mathbf{A}_1}(a_1, b_1) \times \ldots \times \theta^{\mathbf{A}_n}(a_n, b_n).$$

(1)$\Rightarrow$(2). By Proposition 5 we have that $\mathcal{V}$ has the congruence extension property. Also we note that (i) and (1) imply that $\mathcal{V}$ has the Fraser-Horn property.

(2)$\Rightarrow$(1). Let $\mathcal{L}$ be the language of $\mathcal{V}$. We will use the following well known fact.

(ii) If $\sigma : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then $(a, b) \in \theta^{\mathbf{A}}(c, d)$ implies $(\sigma(a), \sigma(b)) \in \theta^{\mathbf{B}}(\sigma(c), \sigma(d))$.

For each $\mathbf{A} \in \mathcal{V}$, let

$$R^{\mathbf{A}} = \{(a, b, c, d) : (a, b) \in \theta^{\mathbf{A}}(c, d)\},$$

and define

$$\mathcal{K} = \{(\mathbf{A}, R^{\mathbf{A}}) : \mathbf{A} \in \mathcal{V}\}.$$

Since $\mathcal{V}$ has definable principal congruences, $\mathcal{K}$ is a first order class and hence $\mathbb{P}_u(\mathcal{K}) \subseteq \mathcal{K}$. Since $\mathcal{V}$ has the Fraser-Horn property, (i) says that $\mathbb{P}_{\text{fin}}(\mathcal{K}) \subseteq \mathcal{K}$. Thus, in order to prove that (2) of Theorem 11 holds we need to check that:

(iii) For all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all homomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$ and all $a, b, c, d \in A_0$, we have that $(a, b, c, d) \in R^{\mathbf{A}}$ implies $(\sigma(a), \sigma(b), \sigma(c), \sigma(d)) \in R^{\mathbf{B}}$.

Or, equivalently

(iv) For all $\mathbf{A}, \mathbf{B} \in \mathcal{Q}$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all homomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$ and all $a, b, c, d \in A_0$, we have that $(a, b) \in \theta^{\mathbf{A}}_{\mathcal{Q}}(c, d)$ implies $(\sigma(a), \sigma(b)) \in \theta^{\mathbf{B}}_{\mathcal{Q}}(\sigma(c), \sigma(d))$.

Since $\mathcal{V}$ has the congruence extension property we can replace in (iv) the occurrence of "$(a, b) \in \theta^{\mathbf{A}}(c, d)$" by "$(a, b) \in \theta^{\mathbf{A}_0}(c, d)$". Hence (iv) follows from (ii). ■

**Corollary 16** *A locally finite variety with the congruence extension property and the Fraser-Horn property has equationally definable principal congruences.*

**Proof.** Combine Corollary 6 with Proposition 15. ■

It is worth mentioning that in the terminology of [5], a variety is said to have equationally definable principal congruences if there is a formula of the form $\exists \bigwedge p = q$ which defines the principal congruences. Thus, Theorem 4 of [5] is not in contradiction with Proposition 15, in fact it coincides with Corollary 24 of the next section.

## Two translation results

As another application of Theorem 11, we obtain a model theoretic proof of the following translation result.

**Proposition 17 ([9, Thm 2.3])** *Let $\mathcal{K}$ be a universal class of $\mathcal{L}$-algebras such that $\mathcal{K} \subseteq \mathcal{Q}_{RFSI}$, for some relatively congruence distributive quasivariety $\mathcal{Q}$. Let $\varphi(\vec{x}) \in [\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$. Then there are $\mathcal{L}$-terms $p_i, q_i$, $i = 1, \ldots, r$ such that*

$$\mathcal{K} \vDash \varphi(\vec{x}) \leftrightarrow \left( \bigwedge_{i=1}^{r} p_i(\vec{x}) = q_i(\vec{x}) \right).$$

**Proof.** We start by proving the proposition for the formula

$$\varphi = (x_1 = x_2 \vee x_3 = x_4).$$

Given $\mathbf{A} \in \mathcal{K}$, let

$$R^{\mathbf{A}} = \{\vec{a} \in A^4 : \mathbf{A} \vDash \varphi(\vec{a})\} = \{(a, b, c, d) \in A^4 : a = b \text{ or } c = d\}.$$

Define $\mathcal{K}' = \{(\mathbf{A}, R^{\mathbf{A}}) : \mathbf{A} \in \mathcal{K}\}$. Note that $\mathcal{K}'$ is universal. We aim to apply (2) of Theorem 11, so we need to show that:

- For all $\mathbf{A} \in \mathbb{P}_u\mathbb{P}_{\mathrm{fin}}(\mathcal{K}')$, all $\mathbf{B} \in \mathbb{P}_u(\mathcal{K}')$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all homomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, and all $a, b, c, d \in A_0$, we have that $(a, b, c, d) \in R^{\mathbf{A}}$ implies $(\sigma(a), \sigma(b), \sigma(c), \sigma(d)) \in R^{\mathbf{B}}$.

Since $\mathcal{K}'$ is universal, we can suppose that $\mathbf{B} \in \mathcal{K}'$, $\mathbf{B}_0 = \mathbf{B}_{\mathcal{L}}$ and $\sigma$ is onto. Also, as $\mathbb{P}_u\mathbb{P}_{\mathrm{fin}}(\mathcal{K}') \subseteq \mathbb{ISPP}_u(\mathcal{K}') \subseteq \mathbb{ISP}(\mathcal{K}')$, we may assume that $\mathbf{A} \leq \Pi\{\mathbf{A}_i : i \in I\}$ is a subdirect product with each $\mathbf{A}_i$ in $\mathcal{K}'$, and that $\mathbf{A}_0 = \mathbf{A}_{\mathcal{L}}$. Since

$$\mathbf{A}/\ker\sigma \simeq \mathbf{B} \in \mathcal{K}'$$

and $\mathcal{K} \subseteq \mathcal{Q}_{RFSI}$, we have that $\ker\sigma$ is a meet irreducible element of $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}_{\mathcal{L}})$. So, as $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}_{\mathcal{L}})$ is distributive, we have that $\ker\sigma$ is a meet prime element of $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}_{\mathcal{L}})$. Let $(a, b, c, d) \in R^{\mathbf{A}}$. We prove that $(\sigma(a), \sigma(b), \sigma(c), \sigma(d)) \in R^{\mathbf{B}}$ (i.e., $\sigma(a) = \sigma(b)$ or $\sigma(c) = \sigma(d)$). For $i \in I$ let $\pi_i : \mathbf{A} \to \mathbf{A}_i$ be the canonical projection. Note that $\ker\pi_i \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}_{\mathcal{L}})$ since $\mathbf{A}/\ker\pi_i \simeq \mathbf{A}_i \in \mathcal{K}'$. From $(a, b, c, d) \in R^{\mathbf{A}}$, it follows that either $a(i) = b(i)$ or $c(i) = d(i)$ for every $i \in I$. Thus

$$\bigcap_{(a,b)\in\ker\pi_i} \ker\pi_i \cap \bigcap_{(c,d)\in\ker\pi_i} \ker\pi_i = \Delta^{\mathbf{A}_{\mathcal{L}}} \subseteq \ker\sigma.$$

Since $\ker\sigma$ is meet prime, this implies that either

$$\bigcap_{(a,b)\in\ker\pi_i} \ker\pi_i \subseteq \ker\sigma$$

or

$$\bigcap_{(c,d)\in\ker\pi_i} \ker\pi_i \subseteq \ker\sigma,$$

which implies $\sigma(a) = \sigma(b)$ or $\sigma(c) = \sigma(d)$. This concludes the proof for the case $\varphi = (x_1 = x_2 \vee x_3 = x_4)$. The case in which $\varphi$ is the formula $x_1 = y_1 \vee x_2 = y_2 \vee \ldots \vee x_n = y_n$ can be proved in a similar manner. Now, the general case follows easily. ■

Strenghtening RFSI to RS allows for the translation of any open formula to a conjunction of equations over $\mathcal{K}$. This is proved in [8] using topological arguments. Combining Proposition 17 and Theorem 1 we get a very simple proof.

**Corollary 18** *Let $\mathcal{K}$ be a universal class of $\mathcal{L}$-algebras such that $\mathcal{K} \subseteq \mathcal{Q}_{RS}$, for some relatively congruence distributive quasivariety $\mathcal{Q}$. Let $\varphi(\vec{x}) \in \mathrm{Op}(\mathcal{L})$. Then there are $\mathcal{L}$-terms $p_i, q_i$, $i = 1, \ldots, r$ such that*

$$\mathcal{K} \vDash \varphi(\vec{x}) \leftrightarrow \left( \bigwedge_{i=1}^{r} p_i(\vec{x}) = q_i(\vec{x}) \right).$$

**Proof.** We show first that there is $\delta(x, y) \in [\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$ such that $\mathcal{K} \vDash x \neq y \leftrightarrow \delta(x, y)$. Given $\mathbf{A} \in \mathcal{K}$, let

$$D^{\mathbf{A}} = \{(a, b) \in A^2 : a \neq b\},$$

and define $\mathcal{K}' = \{(\mathbf{A}, D^{\mathbf{A}}) : \mathbf{A} \in \mathcal{K}\}$. Observe that $\mathcal{K}'$ is universal. We want to apply Theorem 1, thus we need to check that:

- For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K}')$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, all homomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, and all $a, b \in A_0$, we have that $a \neq b$ implies $\sigma(a) \neq \sigma(b)$.

But this is easy. Just note that both $\mathbf{A}_0$ and $\mathrm{Im}\,\sigma$ are in $\mathcal{Q}_{RS}$ since $\mathcal{K}$ is universal. So $\ker \sigma \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}_0) = \{\Delta^{\mathbf{A}_0}, \nabla^{\mathbf{A}_0}\}$, and $\ker \sigma \neq \nabla^{\mathbf{A}_0}$ because $\mathrm{Im}\,\sigma$ is simple and thus non-trivial. It follows that $\sigma$ is one-one. By Theorem 1 we have a formula in $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$ that defines $D$ in $\mathcal{K}$.

Now Proposition 17 produces $\varepsilon\,(x, y) \in [\bigwedge \mathrm{At}(\mathcal{L})]$ such that

$$\mathcal{K} \vDash x \neq y \leftrightarrow \varepsilon(x, y).$$

This, in combination with the fact that disjunctions of equations are equivalent to conjunctions in $\mathcal{K}$ (again by Proposition 17), lets us translate any open formula to a conjunction of equations over $\mathcal{K}$. ■

The translation results above produce the following interesting corollaries for a finite algebra.

**Corollary 19** *Suppose $\mathbf{A}$ is a finite $\mathcal{L}$-algebra such that $\mathbb{S}(\mathbf{A}) \subseteq \mathcal{Q}_{RFSI}$ (resp. $\mathbb{S}(\mathbf{A}) \subseteq \mathcal{Q}_{RS}$), for some relatively congruence distributive quasivariety $\mathcal{Q}$. Let $f : A^n \to A$. The following are equivalent:*

*(1) There is a formula in $[\bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{A}$.*

*(2) For all $\mathbf{S}_1, \mathbf{S}_2 \leq \mathbf{A}$, all homomorphisms (resp. isomorphisms) $\sigma : \mathbf{S}_1 \to \mathbf{S}_2$, and all $a_1, \ldots, a_n \in S_1$ such that $f^{\mathbf{A}}(\vec{a}) \in S_1$, we have $\sigma(f^{\mathbf{A}}(\vec{a})) = f^{\mathbf{A}}(\sigma(a_1), \ldots, \sigma(a_n))$.*

**Proof.** (1)⇒(2). Apply (1)⇒(2) of Theorem 13.

(2)⇒(1). By Corollary 4 there is a formula $\varphi(\vec{x}, z) \in [\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$ (resp. $\varphi(\vec{x}, z) \in \mathrm{Op}(\mathcal{L})$) which defines $f$ in $\mathbb{IS}(\mathbf{A})$. Now use Proposition 17 (resp. Corollary 18) to obtain a conjunction of equations equivalent to $\varphi$ over $\mathbb{IS}(\mathbf{A})$. ■

**Corollary 20** *Suppose $\mathbf{A}$ is a finite $\mathcal{L}$-algebra such that $\mathbb{S}(\mathbf{A}) \subseteq \mathcal{Q}_{RFSI}$ (resp. $\mathbb{S}(\mathbf{A}) \subseteq \mathcal{Q}_{RS}$), for some relatively congruence distributive quasivariety $\mathcal{Q}$. Let $R \subseteq A^n$. The following are equivalent:*

*(1) There is a formula in $[\bigwedge \mathrm{At}(\mathcal{L})]$ which defines $R$ in $\mathbf{A}$.*

*(2) For all $\mathbf{S}_1, \mathbf{S}_2 \leq \mathbf{A}$, all homomorphisms (resp. isomorphisms) $\sigma : \mathbf{S}_1 \to \mathbf{S}_2$, and all $a_1, \ldots, a_n \in S_1$, we have that $(a_1, \ldots, a_n) \in R$ implies $(\sigma(a_1), \ldots, \sigma(a_n)) \in R$.*

# 5 Definability by existential formulas

**Lemma 21** *Let $\mathbf{A}, \mathbf{B}$ be $\mathcal{L}$-structures. Suppose for every sentence $\varphi \in [\exists \bigwedge \mathrm{At}(\mathcal{L})]$ (resp. $\varphi \in [\exists \bigwedge \pm\mathrm{At}(\mathcal{L})]$) we have that $\mathbf{A} \vDash \varphi$ implies $\mathbf{B} \vDash \varphi$. Then there is a homomorphism (resp. embedding) from $\mathbf{A}$ into an ultrapower of $\mathbf{B}$.*

**Proof.** Let $\mathcal{L}_A = \mathcal{L} \cup A$, where each element of $A$ is added as a new constant symbol. Define

$$\Delta = \{\alpha(\vec{a}) : \alpha \in [\bigwedge \mathrm{At}(\mathcal{L})] \text{ and } \mathbf{A} \vDash \alpha(\vec{a})\},$$

i.e., $\Delta$ is the positive atomic diagram of $\mathbf{A}$. Let $I = \mathcal{P}_{\mathrm{fin}}(\Delta)$, and observe that for every $i \in I$ there is an expansion $\mathbf{B}_i$ of $\mathbf{B}$ to $\mathcal{L}_A$ such that $\mathbf{B}_i \vDash i$. Now take an ultrafilter $u$ over $I$ such that for each $i \in I$ the set $\{j \in I : i \subseteq j\}$ is in $u$. Let $\mathbf{U} = \prod \mathbf{B}_i / u$, and notice that $\mathbf{U} \vDash \Delta$. Thus $a \mapsto a^{\mathbf{U}}$ is an homomorphism from $\mathbf{A}$ into $\mathbf{U}_{\mathcal{L}} = \mathbf{B}^I / u$.

The same proof works for the embedding version of the lemma by taking $\Delta = \{\alpha(\vec{a}) : \alpha \in [\pm \bigwedge \mathrm{At}(\mathcal{L})] \text{ and } \mathbf{A} \vDash \alpha(\vec{a})\}$. ■

**Theorem 22** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $R \in \mathcal{L}' - \mathcal{L}$ be an $n$-ary relation symbol. Let $\mathcal{K}$ be any class of $\mathcal{L}'$-structures. Then*

*(1) The following are equivalent:*

*(a) There is a formula in $[\exists \mathrm{Op}(\mathcal{L})]$ which defines $R$ in $\mathcal{K}$.*

*(b) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$ and all embeddings $\sigma : \mathbf{A}_{\mathcal{L}} \to \mathbf{B}_{\mathcal{L}}$, we have that $\sigma : (\mathbf{A}_{\mathcal{L}}, R^{\mathbf{A}}) \to (\mathbf{B}_{\mathcal{L}}, R^{\mathbf{B}})$ is a homomorphism.*

*(2) The following are equivalent:*

*(a) There is a formula in* $[\exists \mathrm{OpHorn}(\mathcal{L})]$ *which defines* $R$ *in* $\mathcal{K}$*.*

*(b) For all* $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u \mathbb{P}_{\mathrm{fin}}(\mathcal{K})$ *and all embeddings* $\sigma : \mathbf{A}_{\mathcal{L}} \to \mathbf{B}_{\mathcal{L}}$*, we have that* $\sigma : (\mathbf{A}_{\mathcal{L}}, R^{\mathbf{A}}) \to (\mathbf{B}_{\mathcal{L}}, R^{\mathbf{B}})$ *is a homomorphism.*

*(3) The following are equivalent:*

*(a) There is a formula in* $[\exists \bigvee \bigwedge \mathrm{At}(\mathcal{L})]$ *which defines* $R$ *in* $\mathcal{K}$*.*

*(b) For all* $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$ *and all homomorphisms* $\sigma : \mathbf{A}_{\mathcal{L}} \to \mathbf{B}_{\mathcal{L}}$*, we have that* $\sigma : (\mathbf{A}_{\mathcal{L}}, R^{\mathbf{A}}) \to (\mathbf{B}_{\mathcal{L}}, R^{\mathbf{B}})$ *is a homomorphism.*

*(4) The following are equivalent:*

*(a) There is a formula in* $[\exists \bigwedge \mathrm{At}(\mathcal{L})]$ *which defines* $R$ *in* $\mathcal{K}$*.*

*(b) For all* $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u \mathbb{P}_{\mathrm{fin}}(\mathcal{K})$ *and all homomorphisms* $\sigma : \mathbf{A}_{\mathcal{L}} \to \mathbf{B}_{\mathcal{L}}$*, we have that* $\sigma : (\mathbf{A}_{\mathcal{L}}, R^{\mathbf{A}}) \to (\mathbf{B}_{\mathcal{L}}, R^{\mathbf{B}})$ *is a homomorphism.*

*Moreover, if modulo isomorphism,* $\mathcal{K}$ *is a finite class of finite structures, then we can remove the operator* $\mathbb{P}_u$ *from the (b) items.*

**Proof.** (1). (a)⇒(b). Note that if $\varphi(\vec{x}) \in [\exists \mathrm{Op}(\mathcal{L})]$ defines $R$ in $\mathcal{K}$, then $\varphi$ defines $R$ in $\mathbb{P}_u(\mathcal{K})$ as well. Now use that $\varphi$ is preserved by embeddings.

(b)⇒(a). For $\mathbf{A} \in \mathbb{P}_u(\mathcal{K})$ and $\vec{a} \in R^{\mathbf{A}}$, let

$$\Gamma^{\vec{a},\mathbf{A}} = \{\alpha(\vec{x}) : \alpha \in \left[\exists \bigwedge \pm\mathrm{At}(\mathcal{L})\right] \text{ and } \mathbf{A} \vDash \alpha(\vec{a})\}.$$

Suppose $\mathbf{B} \in \mathbb{P}_u(\mathcal{K})$ and $\mathbf{B} \vDash \Gamma^{\vec{a},\mathbf{A}}(\vec{b})$. We claim that $\vec{b} \in R^{\mathbf{B}}$. Let $\tilde{\mathcal{L}}$ be the result of adding $n$ new constant symbols to $\mathcal{L}$. Note that every sentence of $\left[\exists \bigwedge \pm\mathrm{At}(\tilde{\mathcal{L}})\right]$ which holds in $(\mathbf{A}_{\mathcal{L}}, \vec{a})$ holds in $(\mathbf{B}_{\mathcal{L}}, \vec{b})$, which by Lemma 21 says that there is an embedding from $(\mathbf{A}_{\mathcal{L}}, \vec{a})$ into an ultrapower $(\mathbf{B}_{\mathcal{L}}, \vec{b})^I/u$. Since $\mathbb{P}_u\mathbb{P}_u(\mathcal{K}) \subseteq \mathbb{P}_u(\mathcal{K})$, (b) says that $(b_1/u, \dots, b_n/u) \in R^{\mathbf{B}^I/u}$, which yields $\vec{b} \in R^{\mathbf{B}}$. So we have that

$$\mathbb{P}_u(\mathcal{K}) \vDash \left( \bigvee_{\mathbf{A} \in \mathbb{P}_u(\mathcal{K}),\ \vec{a} \in R^{\mathbf{A}}} \bigwedge_{\alpha \in \Gamma^{\vec{a},\mathbf{A}}} \alpha(\vec{x}) \right) \leftrightarrow R(\vec{x}),$$

and compactness produces the formula.

(2). (a)⇒(b). Observe that if $\varphi(\vec{x}) \in [\exists \mathrm{OpHorn}(\mathcal{L})]$ defines $R$ in $\mathcal{K}$, then $\varphi$ also defines $R$ in $\mathbb{P}_u\mathbb{P}_{\mathrm{fin}}(\mathcal{K})$. Next use that $\varphi$ is preserved by embeddings.

(b)⇒(a). By (1) we have that there is $\varphi \in [\exists \mathrm{Op}(\mathcal{L})]$ which defines $R$ in $\mathbb{P}_{\mathrm{fin}}(\mathcal{K})$. Now we can apply a similar argument to that used in the proof of Theorem 9, to extract from $\varphi$ a formula of $[\exists \mathrm{OpHorn}(\mathcal{L})]$ which defines $R$ in $\mathcal{K}$.

(3). (b)⇔(a). This is similar to (b)⇔(a) of (1).

(4). (a)⇒(b). Analogous to (a)⇒(b) of (2).

(b)⇒(a). Note that (b) holds also when $\mathbf{B}$ is in $\mathbb{P}\mathbb{P}_u(\mathcal{K})$. Since $\mathbb{P}_u\mathbb{P}_{\text{fin}}(\mathcal{K}) \subseteq \mathbb{P}\mathbb{P}_u(\mathcal{K})$, applying (b)⇒(a) of (3) to the class $\mathbb{P}_{\text{fin}}(\mathcal{K})$ we have that there is a formula $\varphi \in [\exists \bigvee \bigwedge \text{At}(\mathcal{L})]$ which defines $R$ in $\mathbb{P}_{\text{fin}}(\mathcal{K})$. Now we can apply a similar argument to that used in the proof of Theorem 9, to extract from $\varphi$ a formula of $[\exists \bigwedge \text{At}(\mathcal{L})]$ which defines $R$ in $\mathcal{K}$.

The moreover part is left to the reader (see the proof of 7). ■

We state without proof the functional version of the above theorem.

**Theorem 23** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $f_1, \ldots, f_m \in \mathcal{L}' - \mathcal{L}$ be $n$-ary function symbols. Let $\mathcal{K}$ be a class of $\mathcal{L}'$-structures. Then we have:*

*(1) The following are equivalent:*

- *(a) There is a formula in $[\exists \text{Op}(\mathcal{L})]$ which defines $\vec{f}$ in $\mathcal{K}$.*
- *(b) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$ and all embeddings $\sigma : \mathbf{A}_{\mathcal{L}} \to \mathbf{B}_{\mathcal{L}}$, we have that $\sigma : (\mathbf{A}_{\mathcal{L}}, f_1^{\mathbf{A}}, \ldots, f_m^{\mathbf{A}}) \to (\mathbf{B}_{\mathcal{L}}, f_1^{\mathbf{B}}, \ldots, f_m^{\mathbf{B}})$ is an embedding.*

*(2) The following are equivalent:*

- *(a) There is a formula in $[\exists \text{OpHorn}(\mathcal{L})]$ which defines $\vec{f}$ in $\mathcal{K}$.*
- *(b) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u\mathbb{P}_{\text{fin}}(\mathcal{K})$ and all embeddings $\sigma : \mathbf{A}_{\mathcal{L}} \to \mathbf{B}_{\mathcal{L}}$, we have that $\sigma : (\mathbf{A}_{\mathcal{L}}, f_1^{\mathbf{A}}, \ldots, f_m^{\mathbf{A}}) \to (\mathbf{B}_{\mathcal{L}}, f_1^{\mathbf{B}}, \ldots, f_m^{\mathbf{B}})$ is an embedding.*

*(3) The following are equivalent:*

- *(a) There is a formula in $[\exists \bigvee \bigwedge \text{At}(\mathcal{L})]$ which defines $\vec{f}$ in $\mathcal{K}$.*
- *(b) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$ and all homomorphisms $\sigma : \mathbf{A}_{\mathcal{L}} \to \mathbf{B}_{\mathcal{L}}$, we have that $\sigma : (\mathbf{A}_{\mathcal{L}}, f_1^{\mathbf{A}}, \ldots, f_m^{\mathbf{A}}) \to (\mathbf{B}_{\mathcal{L}}, f_1^{\mathbf{B}}, \ldots, f_m^{\mathbf{B}})$ is a homomorphism.*

*(4) The following are equivalent:*

- *(a) There is a formula in $[\exists \bigwedge \text{At}(\mathcal{L})]$ which defines $\vec{f}$ in $\mathcal{K}$.*
- *(b) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u\mathbb{P}_{\text{fin}}(\mathcal{K})$ and all homomorphisms $\sigma : \mathbf{A}_{\mathcal{L}} \to \mathbf{B}_{\mathcal{L}}$, we have that $\sigma : (\mathbf{A}_{\mathcal{L}}, f_1^{\mathbf{A}}, \ldots, f_m^{\mathbf{A}}) \to (\mathbf{B}_{\mathcal{L}}, f_1^{\mathbf{B}}, \ldots, f_m^{\mathbf{B}})$ is a homomorphism.*

*Moreover, if modulo isomorphism $\mathcal{K}$ is a finite class of finite structures, then we can remove the operator $\mathbb{P}_u$ from the (b) items.*

For the case in which $\mathcal{K} = \{\mathbf{A}\}$, with $\mathbf{A}$ finite, (1) and (4) of Theorem 23 are proved in [13].

Using (b)⇒(a) of (4) in Theorem 22 we can prove the following result of [5] (see the paragraph below Corollary 16).

**Corollary 24** *Let $\mathcal{V}$ be a variety with definable principal congruences. Let $\mathcal{L}$ be the language of $\mathcal{V}$. The following are equivalent:*

*(1) There is a formula in $[\exists \bigwedge \text{At}(\mathcal{L})]$ which defines the principal congruences in $\mathcal{V}$.*

*(2) $\mathcal{V}$ has the Fraser-Horn property.*

## Primitive positive functions

Functions definable in a finite algebra $\mathbf{A}$ by a formula of the form $\exists \bigwedge p = q$ are called *primitive positive functions* and they have been extensively studied. For the case in which $\mathcal{K} = \{\mathbf{A}\}$ for some finite algebra $\mathbf{A}$, (4) of Theorem 23 is a well known result [10]. The translation results of Section 4 produce the following.

**Proposition 25** *Suppose $\mathbf{A}$ is a finite $\mathcal{L}$-algebra such that $\mathbb{S}(\mathbf{A}) \subseteq Q_{RFSI}$ (resp. $\mathbb{S}(\mathbf{A}) \subseteq Q_{RS}$), for some relatively congruence distributive quasivariety $Q$. Let $f : A^n \to A$. The following are equivalent:*

*(1) $f$ is primitive positive.*

*(2) If $\sigma : \mathbf{A} \to \mathbf{A}$ is a homomorphism (resp. isomorphism), then $\sigma : (\mathbf{A}, f) \to (\mathbf{A}, f)$ is a homomorphism (resp. isomorphism).*

**Proof.** (2)⇒(1) By (3) (resp. (1)) of Theorem 23 we have that there is a formula $\exists \vec{u}\ \psi(\vec{u}, \vec{x}, z)$, with $\psi \in [\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$ (resp. $[\mathrm{Op}(\mathcal{L})]$), which defines $f$ in $\mathbf{A}$. Since $\mathbb{IS}(\mathbf{A})$ is a universal class, Proposition 17 (resp. Corollary 18) says that there is a $\varphi \in [\bigwedge \mathrm{At}(\mathcal{L})]$ equivalent with $\psi$ over $\mathbb{IS}(\mathbf{A})$. Clearly $\exists \vec{u}\ \varphi(\vec{u}, \vec{x}, z)$ defines $f$ in $\mathbf{A}$. ■

### 5.0.1 Primitive positive functions in Stone algebras

As an application of the results in Section 4 and the current section we characterize primitive positive functions and functions definable by formulas of the form $\bigwedge p = q$ in Stone algebras. If $\mathbf{L}$ is a Stone algebra, let $\to^{\mathbf{L}}$ denote its Heyting implication, when it does exist. A *three valued Heyting algebra* is a Heyting algebra belonging to the variety generated by the three element Heyting algebra.

**Proposition 26** *Let $\mathbf{L} = (L, \vee, \wedge, ^*, 0, 1)$ be a Stone algebra and let $f : L^n \to L$ be any function. Let $\mathcal{L} = \{\vee, \wedge, ^*, 0, 1\}$. Then:*

*(1) If the Heyting implication exists in $\mathbf{L}$, and $(L, \vee, \wedge, \to^{\mathbf{L}}, 0, 1)$ is a three valued Heyting algebra, then the following are equivalent:*

  *(a) There is a formula in $[\bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{L}$.*

  *(b) There is a formula in $[\exists \bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{L}$.*

  *(c) There is a term of the language $\{\vee, \wedge, \to, ^*, 0, 1\}$ which represents $f$ in $\mathbf{L}$.*

*(2) If $\mathbf{L}$ does not satisfy the hypothesis of (1), then the following are equivalent:*

  *(a) There is a formula in $[\bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{L}$.*

  *(b) There is a formula in $[\exists \bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{L}$.*

  *(c) There is an $\mathcal{L}$-term which represents $f$ in $\mathbf{L}$.*

**Proof.** Let $\mathbf{3}$ be the three element Stone algebra $(\{0,1/2,1\},\max,\min,^*,0,1)$. Let $\mathbf{2}$ denote the subalgebra of $\mathbf{3}$ with universe $\{0,1\}$, i.e. $\mathbf{2}$ is the two element Boolean algebra. First we prove a series of claims.

**Claim 1.** Let $f:\{0,1/2,1\}^n \to \{0,1/2,1\}$. The following are equivalent:

(i) There is a formula in $[\bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{3}$.

(ii) There is a formula in $[\exists \bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{3}$.

(iii) $f$ is a term function of $(\mathbf{3},\to^{\mathbf{3}})$.

Proof. Since $^{**}:\{0,1/2,1\}\to\{0,1/2,1\}$ is the only non trivial homomorphism between subalgebras of $\mathbf{3}$, Corollary 19 and Proposition 25 say that both (i) and (ii) are equivalent to

(iv) $f(x_1,\ldots,x_n)^{**}=f(x_1^{**},\ldots,x_n^{**})$, for any $x_1,\ldots,x_n\in\{0,1/2,1\}$.

Since $^{**}:\{0,1/2,1\}\to\{0,1/2,1\}$ is a Heyting homomorphism, we have that (iii) implies (iv), and so (iii) implies (i) and (ii). Suppose that (iv) holds and that $f$ is not a term function of $(\mathbf{3},\to^{\mathbf{3}})$. We will arrive at a contradiction. Note that (iv) implies that $\{0,1\}$ is closed under $f$. Since $f$ is not a term function, the Baker-Pixley Theorem says that at least one of the following subuniverses of $(\mathbf{3},\to^{\mathbf{3}})\times(\mathbf{3},\to^{\mathbf{3}})$ is not closed under $f\times f$,

$$\begin{aligned}
S_1 &= \{(0,0),(1/2,1/2),(1,1/2),(1/2,1),(1,1)\},\\
S_2 &= \{(0,0),(1/2,1),(1,1)\},\\
S_3 &= \{(0,0),(1,1/2),(1,1)\}.
\end{aligned}$$

(The other subuniverses of $(\mathbf{3},\to^{\mathbf{3}})\times(\mathbf{3},\to^{\mathbf{3}})$ are clearly closed under $f\times f$.) Suppose $S_1$ is not closed under $f\times f$. Note that $\mathbf{S}_1$ is generated by $\{(1,1/2),(1/2,1)\}$. So, we can suppose that $f$ is binary, satisfies (iv) and $(f(1,1/2),f(1/2,1))\notin S_1$. If $(f(1,1/2),f(1/2,1))=(0,1/2)$, then

$$\begin{aligned}
(0,1) &= (0^{**},1/2^{**})\\
&= (f(1^{**},1/2^{**}),f(1/2^{**},1^{**}))\\
&= (f(1,1),f(1,1))
\end{aligned}$$

which is absurd. The other cases are similar. If either $S_2$ or $S_3$ is not closed under $f\times f$ we can arrive to a contradiction in a similar manner.

**Claim 2.** Let $f:\{0,1/2,1\}^n\to\{0,1/2,1\}$. If there is a formula $\varphi\in[\exists\bigwedge\mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{3}$ and $f$ is not a term function of $\mathbf{3}$, then $\to^{\mathbf{3}}$ is a term function of $(\mathbf{3},f)$.

Proof. By Claim 1, $f$ is a term function of $(\mathbf{3},\to^{\mathbf{3}})$. By the Baker-Pixley Theorem, since $f$ is not a term function of $\mathbf{3}$ we have that

$$S_4=\{(0,0),(1/2,1/2),(1/2,1),(1,1)\}$$

or

$$S_5 = \{(0,0),(1/2,1/2),(1,1/2),(1,1)\}$$

is not closed under $f \times f$ (the other subuniverses are Heyting subuniverses and hence they are closed under $f \times f$). But $\mathbf{S}_4$ and $\mathbf{S}_5$ are isomorphic and $f \times f$ is defined by $\varphi$ in $\mathbf{3} \times \mathbf{3}$ which implies that both $S_4$ and $S_5$ are not closed under $f \times f$. Thus every subalgebra of $(\mathbf{3}, f) \times (\mathbf{3}, f)$ is a Heyting subalgebra which by the Baker-Pixley Theorem says that $\rightarrow^{\mathbf{3}}$ is a term operation of $(\mathbf{3}, f)$.

**Claim 3.** If $\mathbf{L}$ is a Stone algebra which is a subdirect product of a family of Stone algebras $\{\mathbf{L}_i : i \in I\}$, and $\varphi \in [\exists \bigwedge \mathrm{At}(\mathcal{L})]$ defines an $n$-ary function $f$ on $L$, then $\varphi$ defines a function $f_i$ on each $L_i$, and $f$ is the restriction of $(f_i)_{i \in I}$ to $L^n$.

Proof. Since $\mathbf{L}_i$ is a homomorphic image of $\mathbf{L}$, we have that $\mathbf{L}_i \vDash \exists z \varphi(\vec{x}, z)$. Suppose $\varphi(\vec{x}, z) = \exists \vec{w} \psi(\vec{w}, \vec{x}, z)$ with $\psi \in [\bigwedge \mathrm{At}(\mathcal{L})]$. Since every subquasivariety of the variety of Stone algebras is a variety and $\mathbf{L} \vDash \psi(\vec{w}, \vec{x}, z_1) \wedge \psi(\vec{w}, \vec{x}, z_2) \rightarrow z_1 = z_2$, we have that $\mathbf{L}_i \vDash \varphi(\vec{x}, z_1) \wedge \varphi(\vec{x}, z_2) \rightarrow z_1 = z_2$, which says that $\varphi$ defines a function on $L_i$.

**Claim 4.** If $\mathbf{A} \leq \mathbf{B}$, and $\varphi \in [\exists \bigwedge \mathrm{At}(\mathcal{L})]$ defines $n$-ary functions $f$ on $\mathbf{A}$ and $g$ on $\mathbf{B}$, then $f$ is equal to the restriction of $g$ to $A^n$.

Proof. Trivial.

We are ready to prove (1). Suppose the Heyting implication exists in $\mathbf{L}$, and $(L, \vee, \wedge, \rightarrow^{\mathbf{L}}, 0, 1)$ is a three valued Heyting algebra. Recall that $(\{0,1\}, \max, \min, \rightarrow^{\mathbf{2}}, 0, 1)$ and $(\{0, 1/2, 1\}, \max, \min, \rightarrow^{\mathbf{3}}, 0, 1)$ are the only subdirectly irreducible three valued Heyting algebras. Since $x^* = x \rightarrow^{\mathbf{L}} 0$ for every $x \in L$, we can suppose that

$$(L, \vee, \wedge, ^*, \rightarrow^{\mathbf{L}}, 0, 1) \leq \prod \{\mathbf{L}_u : u \in I \cup J\}$$

is a subdirect product, where $\mathbf{L}_u = (\mathbf{2}, \rightarrow^{\mathbf{2}})$ for $u \in I$, and $\mathbf{L}_u = (\mathbf{3}, \rightarrow^{\mathbf{3}})$ for $u \in J$. Suppose that $J \neq \emptyset$ and let $u_0 \in J$. The case $J = \emptyset$ is left to the reader.

(a)⇒(b). This is clear.

(b)⇒(c). By Claim 3, $\varphi$ defines a function $f_u$ on each $\mathbf{L}_u$ and $f$ is the restriction of $(f_u)_{u \in I \cup J}$ to $L^n$. Note that $f_u = f_v$ whenever $u, v \in I$ or $u, v \in J$. By Claim 4, for every $u \in I$, the function $f_u$ is the restriction of $f_{u_0}$ to $\{0,1\}^n$. By Claim 1, there is a $\{\vee, \wedge, ^*, \rightarrow, 0, 1\}$-term $p$ such that $f_{u_0} = p^{\mathbf{L}_{u_0}}$. Note that $f_u = p^{\mathbf{L}_u}$, for every $u \in I \cup J$ and hence $f = p^{(L, \vee, \wedge, ^*, \rightarrow^{\mathbf{L}}, 0, 1)}$.

(c)⇒(a). Suppose $p$ is a $\{\vee, \wedge, ^*, \rightarrow, 0, 1\}$-term such that $f = p^{(L, \vee, \wedge, ^*, \rightarrow^{\mathbf{L}}, 0, 1)}$. By Claim 1, $p^{\mathbf{L}_{u_0}}$ is definable in $\mathbf{L}_{u_0} = (\mathbf{3}, \rightarrow^{\mathbf{3}})$ by a formula $\varphi \in [\bigwedge \mathrm{At}(\mathcal{L})]$. Note that, for every $u \in I \cup J$, we have that $p^{\mathbf{L}_u}$ is defined in $\mathbf{L}_u$ by $\varphi$. But $f$ is the restriction of $(p^{\mathbf{L}_u})_{u \in I \cup J}$ to $L$, which implies that $f$ is defined by $\varphi$ in $\mathbf{L}$.

Next we prove (2). The implication (c)⇒(a) is immediate. In fact, if an $\mathcal{L}$-term $t(\vec{x})$ represents $f$ in $\mathbf{L}$, then the formula $z_1 = t(\vec{x})$ defines $f$ in $\mathbf{L}$. To show (b)⇒(c) we prove:

**Claim 5.** Assume $f : L^n \to L$ is not representable by an $\mathcal{L}$-term in $\mathbf{L}$, and suppose $f$ is defined in $\mathbf{L}$ by a formula $\varphi \in [\exists \bigwedge \mathrm{At}(\mathcal{L})]$. Then the Heyting implication exists in $\mathbf{L}$, and $(L, \vee, \wedge, \to^{\mathbf{L}}, 0, 1)$ is a three valued Heyting algebra.

To prove this claim we first note that since $\mathbf{2}$ and $\mathbf{3}$ are the only subdirectly irreducible Stone algebras we can suppose that

$$\mathbf{L} \leq \prod \{\mathbf{L}_u : u \in I \cup J\}$$

is a subdirect product, where $\mathbf{L}_u = \mathbf{2}$ for $u \in I$, and $\mathbf{L}_u = \mathbf{3}$ for $u \in J$. By Claim 3, $\varphi$ defines a function $f_u$ on each $\mathbf{L}_u$ and $f$ is the restriction of $(f_u)_{u \in I \cup J}$ to $L$. Note that $f_u = f_v$ whenever $u, v \in I$ or $u, v \in J$. Also, by Claim 4, the function $f_u$ is the restriction of $f_v$ to $\{0, 1\}$, whenever $u \in I$ and $v \in J$. If $J = \emptyset$, then since $\mathbf{2}$ is primal, there is an $\mathcal{L}$-term $p$ which represents $f_u$ in $\mathbf{2}$. But this is impossible since this implies that $f$ is representable by $p$ in $\mathbf{L}$. So $J \neq \emptyset$. Let $u_0 \in J$. Since $f$ is not representable by an $\mathcal{L}$-term in $\mathbf{L}$, we have that $f_{u_0}$ is not representable by an $\mathcal{L}$-term in $\mathbf{3}$. Thus Claim 2 implies that $\to^{\mathbf{3}}$ is a term function of $(\mathbf{3}, f_{u_0})$. Note that the same term witnesses that $\to^{\mathbf{L}_u}$ is a term function of $(\mathbf{L}_u, f_u)$ for every $u \in I \cup J$. But $\mathbf{L}$ is closed under $(f_u)_{u \in I \cup J}$, and hence we have that $\mathbf{L}$ is closed under $(\to^{\mathbf{L}_u})_{u \in I \cup J}$. This says that the Heyting implication exists in $\mathbf{L}$, and that $(L, \vee, \wedge, \to^{\mathbf{L}}, 0, 1)$ is a three valued Heyting algebra. ■

Functions definable by a formula of the form $\bigwedge p = q$ are called *mono-algebraic*. They are studied in [7] using sheaf representations.

#### 5.0.2 Primitive positive functions in De Morgan algebras

Let $\mathcal{L} = \{\vee, \wedge, ^-, 0, 1\}$ be the language of De Morgan algebras. Let $\mathbf{M} = (\{0, a, b, 1\}, \vee, \wedge, ^-, 0, 1)$ where $(\{0, a, b, 1\}, \vee, \wedge, 0, 1)$ is the two-atom Boolean lattice and $\bar{0} = 1$, $\bar{1} = 0$, $\bar{a} = a$ and $\bar{b} = b$. It is well known that $\mathbf{M}$ is a simple De Morgan algebra which generates the variety of all De Morgan algebras [1]. Primitive positive functions of $\mathbf{M}$ can be characterized as follows.

**Proposition 27** *For any function $f : M^n \to M$, the following are equivalent*

*(1) There is a formula in $[\bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{M}$.*

*(2) There is a formula in $[\exists \bigwedge \mathrm{At}(\mathcal{L})]$ which defines $f$ in $\mathbf{M}$.*

*(3) $f$ is a term operation of $(\mathbf{M}, ^\circ)$, where $0^\circ = 0$, $1^\circ = 1$, $a^\circ = b$ and $b^\circ = a$.*

*(4) $f(x_1, \ldots, x_n)^\circ = f(x_1^\circ, \ldots, x_n^\circ)$, for any $x_1, \ldots, x_n \in M$.*

**Proof.** (1)⇒(2) This is trivial.

(2)⇒(3) We note that

$$^\circ : \{0, a, b, 1\} \to \{0, a, b, 1\}$$

is an automorphism of $(\mathbf{M},^{\circ})$. Since $f$ is preserved by $^{\circ}$, assumption (2) implies that $\{0,1\}$ is closed under $f$. Also note that

$$\overline{x}^{\circ} = \text{ Boolean complement of } x,$$

which says that $(\mathbf{M},^{\circ})$ generates an arithmetical variety. Since this algebra is simple and $\{0,1\}$ is its only proper subuniverse, Fleischer's theorem says that the subuniverses of $(\mathbf{M},^{\circ})\times(\mathbf{M},^{\circ})$ are: $\{0,1\}\times\{0,1\}$, $M\times M$, $\{(x,x^{\circ}) : x \in M\}$, $\{(x,x) : x \in M\}$ and $\{(0,0),(1,1)\}$. Each of these is easily seen to be closed under $f \times f$, which by the Baker-Pixley Theorem says that $f$ is a term function of $(\mathbf{M},^{\circ})$.

(3)⇒(4) This is clear since $^{\circ}$ is an automorphism of $(\mathbf{M},^{\circ})$.

(4)⇒(1) Note that $\mathbf{M}$ has three non trivial inner isomorphisms which are $^{\circ}$ and the restrictions of $^{\circ}$ to $\{0,a,1\}$ and $\{0,b,1\}$. Since the variety of De Morgan algebras is congruence distributive, Corollary 19 says that (1) holds. ■

# 6 Term interpolation

Given a structure $\mathbf{A}$, an interesting –albeit often elusive– problem is to provide a useful description of its term-operations. That is, to give concise (semantical) conditions that characterize when a given function $f : A^n \to A$ is a term-operation. This is beautifully accomplished in the classical Baker-Pixley Theorem for the case in which $\mathbf{A}$ is finite and has a near-unanimity term [2].

A natural way to generalize this problem to classes of structures is as follows. Given a class $\mathcal{K}$ of $\mathcal{L}$-structures and a map $\mathbf{A} \to f^{\mathbf{A}}$ which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary operation $f^{\mathbf{A}} : A^n \to A$, provide conditions that guarantee the existence of a term $t$ such that $t^{\mathbf{A}} = f^{\mathbf{A}}$ on every $\mathbf{A}$ in $\mathcal{K}$. We address this problem for classes in the current section, and obtain some interesting results including generalizations of the aforementioned Baker-Pixley Theorem and for Pixley's theorem characterizing the term-operations of quasiprimal algebras [14].

Another avenue of generalization we considered are functions that are interpolated by a finite number of terms. This is also looked at in the setting of classes.

### Term-valued functions by cases

Let $f \in \mathcal{L}$ be a function symbol, and let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. Given $\mathcal{L}$-terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$ and first order $\mathcal{L}$-formulas $\varphi_1(\vec{x}), \ldots, \varphi_k(\vec{x})$, we write

$$f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k} \text{ in } \mathcal{K}$$

to express that $\mathcal{K} \vDash \varphi_1(\vec{x}) \vee \cdots \vee \varphi_k(\vec{x})$ and

$$f^{\mathbf{A}}(\vec{a}) = \begin{cases} t_1^{\mathbf{A}}(\vec{a}) & \text{if } \mathbf{A} \vDash \varphi_1(\vec{a}) \\ \vdots & \vdots \\ t_k^{\mathbf{A}}(\vec{a}) & \text{if } \mathbf{A} \vDash \varphi_k(\vec{a}) \end{cases}$$

for all $\vec{a} \in A^n$ and $\mathbf{A} \in \mathcal{K}$. (Note that as $f^{\mathbf{A}}$ is a function the definition by cases is not ambiguous.) We say that a term $t(\vec{x})$ *represents* $f$ in $\mathcal{K}$ if $f^{\mathbf{A}}(\vec{a}) = t^{\mathbf{A}}(\vec{a})$, for all $\mathbf{A} \in \mathcal{K}$ and $\vec{a} \in A^n$.

With the help of results from previous sections it is possible to characterize when $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$, with the $t_i$'s not involving $f$ and a fixed format for the $\varphi_i$'s.

**Theorem 28** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $f \in \mathcal{L}' - \mathcal{L}$ be an $n$-ary function symbol. Let $\mathcal{K}$ be a class of $\mathcal{L}'$-structures. The following are equivalent:*

*(1) $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$ in $\mathcal{K}$, with each $t_i$ an $\mathcal{L}$-term and each $\varphi_i$ in $\mathrm{Op}(\mathcal{L})$.*

*(2) The following conditions hold:*

*(a) For all $\mathbf{A} \in \mathbb{P}_u(\mathcal{K})$ and all $\mathbf{S} \leq \mathbf{A}_{\mathcal{L}}$, we have that $S$ is closed under $f^{\mathbf{A}}$.*

*(b) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, and all isomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, we have that $\sigma : (\mathbf{A}_0, f^{\mathbf{A}}|_{A_0}) \to (\mathbf{B}_0, f^{\mathbf{B}}|_{B_0})$ is an isomorphism.*

*Moreover, if $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $(n+1)$-generated substructures and each one is finite, then we can remove the operator $\mathbb{P}_u$ from (a) and (b).*

**Proof.** (1)$\Rightarrow$(2). If $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$ in $\mathcal{K}$, with each $t_i$ an $\mathcal{L}$-term and each $\varphi_i$ in $\mathrm{Op}(\mathcal{L})$, then $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$ in $\mathbb{P}_u(\mathcal{K})$. Now (a) and (b) are routine verifications.

(2)$\Rightarrow$(1). We first note that (a) implies

$$\mathbb{P}_u(\mathcal{K}) \vDash \bigvee_{t(\vec{x}) \text{ an } \mathcal{L}\text{-term}} f(\vec{x}) = t(\vec{x}),$$

which by compactness says that there are $\mathcal{L}$-terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$ such that

$$\mathbb{P}_u(\mathcal{K}) \vDash f(\vec{x}) = t_1(\vec{x}) \vee \cdots \vee f(\vec{x}) = t_k(\vec{x}).$$

Since (b) holds, Theorem 3 implies that there exists a formula $\varphi \in \mathrm{Op}(\mathcal{L})$ which defines $f$ in $\mathcal{K}$. It is clear that for any $\mathbf{A} \in \mathcal{K}$ we have

$$\mathcal{K} \vDash \varphi(\vec{x}, t_1(\vec{x})) \vee \cdots \vee \varphi(\vec{x}, t_k(\vec{x}))$$

and

$$f^{\mathbf{A}}(\vec{a}) = \begin{cases} t_1^{\mathbf{A}}(\vec{a}) & \text{if } \mathbf{A} \vDash \varphi(\vec{a}, t_1(\vec{a})) \\ \quad\vdots & \qquad\qquad \vdots \\ t_k^{\mathbf{A}}(\vec{a}) & \text{if } \mathbf{A} \vDash \varphi(\vec{a}, t_k(\vec{a})) \end{cases}$$

for all $\vec{a} \in A^n$. So we have proved that $f = t_1|_{\varphi_1} \cup \ \ldots \ \cup t_k|_{\varphi_k}$ in $\mathcal{K}$, where $\varphi_i(\vec{x}) = \varphi(\vec{x}, t_i(\vec{x}))$, $i = 1, \ldots, k$.

If $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $(n+1)$-generated substructures and each one is finite, then we note that there are $\mathcal{L}$-terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$ such that

$$\mathcal{K} \models f(\vec{x}) = t_1(\vec{x}) \vee \cdots \vee f(\vec{x}) = t_k(\vec{x})$$

and the proof can be continued in the same manner as above. ■

**Corollary 29** *Let $\mathcal{K}$ be any class of $\mathcal{L}$-algebras contained in a locally finite variety. Suppose $\mathbf{A} \to f^{\mathbf{A}}$ is a map which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary operation $f^{\mathbf{A}} : A^n \to A$. The following are equivalent:*

*(1)* $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$ *in $\mathcal{K}$, with each $t_i$ an $\mathcal{L}$-term and each $\varphi_i$ in $\mathrm{Op}(\mathcal{L})$.*

*(2) The following conditions hold:*

  *(a) If $\mathbf{S} \leq \mathbf{A} \in \mathcal{K}$, then $S$ is closed under $f^{\mathbf{A}}$.*

  *(b) If $\mathbf{A}_0 \leq \mathbf{A} \in \mathcal{K}$, $\mathbf{B}_0 \leq \mathbf{B} \in \mathcal{K}$ and $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$ is an isomorphism, then $\sigma : (\mathbf{A}_0, f^{\mathbf{A}}) \to (\mathbf{B}_0, f^{\mathbf{B}})$ is an isomorphism.*

Using Theorem 28 and its proof as a blueprint it is easy to produce analogous results for other families of formulas. For example here is the positive open case.

**Theorem 30** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages and let $f \in \mathcal{L}' - \mathcal{L}$ be an $n$-ary function symbol. Let $\mathcal{K}$ be a class of $\mathcal{L}'$-structures. The following are equivalent:*

*(1)* $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$ *in $\mathcal{K}$, with each $t_i$ an $\mathcal{L}$-term and each $\varphi_i$ in* $[\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$.

*(2)* $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$ *in $\mathcal{K}$, with each $t_i$ an $\mathcal{L}$-term and each $\varphi_i$ in* $[\bigwedge \mathrm{At}(\mathcal{L})]$.

*(3) The following conditions hold:*

  *(a) For all $\mathbf{A} \in \mathbb{P}_u(\mathcal{K})$ and all $\mathbf{S} \leq \mathbf{A}_{\mathcal{L}}$, we have that $S$ is closed under $f^{\mathbf{A}}$.*

  *(b) For all $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$, all $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$, and all homomorphisms $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$, we have that $\sigma : (\mathbf{A}_0, f^{\mathbf{A}}) \to (\mathbf{B}_0, f^{\mathbf{B}})$ is a homomorphism.*

*Moreover, if $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $(n+1)$-generated substructures and each one is finite, then we can remove the operator $\mathbb{P}_u$ from (a) and (b).*

## Pixley's theorem for classes

Recall that the *ternary discriminator* on the set $\mathbf{A}$ is the function

$$d^{\mathbf{A}}(x,y,z) = \begin{cases} z & \text{if } x = y, \\ x & \text{if } x \neq y. \end{cases}$$

An algebra $\mathbf{A}$ is called *quasiprimal* if it is finite and has the discriminator as a term function.

A well known result of A. Pixley [14] characterizes quasiprimal algebras as those finite algebras in which every function preserving the inner isomorphisms is a term function. Of course the ternary discriminator preserves the inner isomorphisms and hence one direction of the theorem is trivial. The following theorem generalizes the non trivial direction.

**Theorem 31** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages without relation symbols and let $f \in \mathcal{L}' - \mathcal{L}$ be an $n$-ary function symbol. Let $\mathcal{K}$ be a class of $\mathcal{L}'$-algebras such that there is an $\mathcal{L}$-term representing the ternary discriminator in each member of $\mathcal{K}$. Suppose that*

- *(a) If $\mathbf{A} \in \mathbb{P}_u(\mathcal{K})$ and $\mathbf{S} \leq \mathbf{A}_{\mathcal{L}}$, then $S$ is closed under $f^{\mathbf{A}}$.*
- *(b) If $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$, $\mathbf{A}_0 \leq \mathbf{A}_{\mathcal{L}}$, $\mathbf{B}_0 \leq \mathbf{B}_{\mathcal{L}}$ and $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$ is an isomorphism, then $\sigma : (\mathbf{A}_0, f^{\mathbf{A}}) \to (\mathbf{B}_0, f^{\mathbf{B}})$ is an isomorphism.*

*Then $f$ is representable by an $\mathcal{L}$-term in $\mathcal{K}$. Moreover, if $\mathcal{K}_{\mathcal{L}}$ has finitely many isomorphism types of $(n+1)$-generated subalgebras and each one is finite, then we can remove the operator $\mathbb{P}_u$ from (a) and (b).*

**Proof.** By Theorem 28 we have that $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$ in $\mathcal{K}$, with each $t_i$ an $\mathcal{L}$-term and each $\varphi_i$ in $\mathrm{Op}(\mathcal{L})$. We shall prove that $f$ is representable by an $\mathcal{L}$-term in $\mathcal{K}$. Of course, if $k = 1$, the theorem follows. Suppose $k > 1$. We show that we can reduce $k$.

Let $t(x,y,z)$ be an $\mathcal{L}$-term representing the discriminator in $\mathcal{K}$. Then the $\mathcal{L}$-term

$$D(x,y,z,w) = t\left(t\left(x,y,z\right), t\left(x,y,w\right), w\right)$$

represents the *quaternary discriminator* in $\mathcal{K}$, that is, for every $a,b,c,d \in A$, with $\mathbf{A} \in \mathcal{K}$,

$$D^{\mathbf{A}}(a,b,c,d) = \begin{cases} c & a = b \\ d & a \neq b. \end{cases}$$

Having a discriminator term for $\mathcal{K}$ also provides the following translation property (see [16]):

- For every open $\mathcal{L}$-formula $\varphi(\vec{x})$ there exist $\mathcal{L}$-terms $p(\vec{x})$ and $q(\vec{x})$ such that either $\mathcal{K} \vDash \varphi(\vec{x}) \leftrightarrow p(\vec{x}) = q(\vec{x})$ or $\mathcal{K} \vDash \varphi(\vec{x}) \leftrightarrow p(\vec{x}) \neq q(\vec{x})$.

Now, suppose for example that

$$\mathcal{K} \vDash \varphi_1(\vec{x}) \leftrightarrow p_1(\vec{x}) \neq q_1(\vec{x}),$$
$$\mathcal{K} \vDash \varphi_2(\vec{x}) \leftrightarrow p_2(\vec{x}) \neq q_2(\vec{x}),$$

for some $\mathcal{L}$-terms $p_i(\vec{x})$ and $q_i(\vec{x})$. Then

$$f^{\mathbf{A}}(\overline{a}) = \begin{cases} D(p_1, q_1, t_2, t_1)^{\mathbf{A}}(\vec{a}) & \text{if } \mathbf{A} \vDash \varphi(\vec{a}) \\ t_3^{\mathbf{A}}(\vec{a}) & \text{if } \mathbf{A} \vDash \varphi_3(\vec{a}) \\ \quad\vdots & \qquad\vdots \\ t_k^{\mathbf{A}}(\vec{a}) & \text{if } \mathbf{A} \vDash \varphi_k(\vec{a}) \end{cases}$$

where $\varphi(\vec{x}) = p_1(\vec{x}) \neq q_1(\vec{x}) \vee (p_1(\vec{x}) = q_1(\vec{x}) \wedge p_2(\vec{x}) \neq q_2(\vec{x}))$. The other cases are similar. ■

For the locally finite case Pixley's theorem can be generalized as follows.

**Theorem 32** *Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras contained in a locally finite variety. The following are equivalent:*

*(1) There is an $\mathcal{L}$-term representing the ternary discriminator on each member of $\mathcal{K}$.*

*(2) Assume $\mathbf{A} \to f^{\mathbf{A}}$ is a map which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary operation $f^{\mathbf{A}} : A^n \to A$ in such a manner that:*

*(a) for all $\mathbf{A} \in \mathcal{K}$ and all $\mathbf{S} \leq \mathbf{A}$ we have that $S$ is closed under $f^{\mathbf{A}}$, and*

*(b) for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, all $\mathbf{A}_0 \leq \mathbf{A}$, $\mathbf{B}_0 \leq \mathbf{B}$, and every isomorphism $\sigma : \mathbf{A}_0 \to \mathbf{B}_0$ we have that $\sigma : (\mathbf{A}_0, f^{\mathbf{A}}) \to (\mathbf{B}_0, f^{\mathbf{B}})$ is an isomorphism.*

*Then $f$ is representable by an $\mathcal{L}$-term in $\mathcal{K}$.*

**Proof.** (1)⇒(2). This follows from Theorem 31 applied to the class $\{(\mathbf{A}, f^{\mathbf{A}}) : \mathbf{A} \in \mathcal{K}\}$.

(2)⇒(1). It is clear that the map $\mathbf{A} \to d^{\mathbf{A}}$ which assigns to each $\mathbf{A} \in \mathcal{K}$ the ternary discriminator on $\mathbf{A}$, satisfies (a) and (b) of (2). ■

## Baker-Pixley's theorem for classes

Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras. An $\mathcal{L}$-term $M(x, y, z)$ is called a *majority term for $\mathcal{K}$* if the following identities hold in $\mathcal{K}$

$$M(x, x, y) \approx M(x, y, x) \approx M(y, x, x) \approx x.$$

Next, we shall give a generalization of the well known theorem of Baker and Pixley [2] on the existence of terms representing functions in finite algebras with a majority term. First a lemma.

**Lemma 33** *Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras contained in a locally finite variety. Let $\mathbf{A} \to f^{\mathbf{A}}$ be a map which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary operation $f^{\mathbf{A}} : A^n \to A$. Suppose that for any $m \in \mathbb{N}$, $\mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{K}$ and $\mathbf{S} \leq \mathbf{A}_1 \times \ldots \times \mathbf{A}_m$, we have that $S$ is closed under $f^{\mathbf{A}_1} \times \ldots \times f^{\mathbf{A}_m}$. Then $f$ is representable by an $\mathcal{L}$-term in $\mathcal{K}$.*

**Proof.** Let $t_1(\vec{x}), \ldots, t_k(\vec{x})$ be $\mathcal{L}$-terms such that for every $\mathbf{A}$ in the variety generated by $\mathcal{K}$ and every $\vec{a} \in A^n$ we have that the subalgebra of $\mathbf{A}$ generated by $a_1, \ldots, a_n$ has universe $\{t_1^{\mathbf{A}}(\vec{a}), \ldots, t_k^{\mathbf{A}}(\vec{a})\}$. We prove that $f$ is representable by $t_i$ in $\mathcal{K}$, for some $i$. Suppose to the contrary that for each $i$ there are $\mathbf{A}_i \in \mathcal{K}$ and $\vec{a}^i = (a_1^i, \ldots, a_n^i) \in A_i^n$ such that $f^{\mathbf{A}_i}(\vec{a}^i) \neq t_i^{\mathbf{A}_i}(\vec{a}^i)$. Let $\mathbf{S}$ be the subalgebra of $\mathbf{A}_1 \times \ldots \times \mathbf{A}_k$ generated by $\{p_j : j = 1, \ldots, n\}$, where $p_j = (a_j^1, a_j^2, \ldots, a_j^k)$. Since $S$ is closed under $f^{\mathbf{A}_1} \times \ldots \times f^{\mathbf{A}_k}$ we have that

$$f^{\mathbf{A}_1} \times \ldots \times f^{\mathbf{A}_k}(p_1, \ldots, p_n) = (f^{\mathbf{A}_1}(\vec{a}^1), \ldots, f^{\mathbf{A}_k}(\vec{a}^k)) \in S.$$

Thus there is $i$ such that

$$(f^{\mathbf{A}_1}(\vec{a}^1), \ldots, f^{\mathbf{A}_k}(\vec{a}^k)) = t_i^{\mathbf{A}_1 \times \ldots \times \mathbf{A}_m}(p_1, \ldots, p_n),$$

which produces

$$(f^{\mathbf{A}_1}(\vec{a}^1), \ldots, f^{\mathbf{A}_k}(\vec{a}^k)) = (t_i^{\mathbf{A}_1}(\vec{a}^1), \ldots, t_i^{\mathbf{A}_k}(\vec{a}^k)).$$

In particular we have that $f^{\mathbf{A}_i}(\vec{a}^i) = t_i^{\mathbf{A}_i}(\vec{a}^i)$, a contradiction. ■

**Theorem 34** *Let $\mathcal{K}$ be a class of algebras contained in a locally finite variety and suppose that $\mathcal{K}$ has a majority term. Let $\mathbf{A} \to f^{\mathbf{A}}$ be a map which assigns to each $\mathbf{A} \in \mathcal{K}$ an $n$-ary operation $f^{\mathbf{A}} : A^n \to A$. Assume that for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and every $\mathbf{S} \leq \mathbf{A} \times \mathbf{B}$ we have that $S$ is closed under $f^{\mathbf{A}} \times f^{\mathbf{B}}$. Then $f$ is representable by a term in $\mathcal{K}$.*

**Proof.** First we show that:

(i) If $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, $\vec{a} \in A^n$ and $\vec{b} \in B^n$, then there is a term $t(\vec{x})$ satisfying $t^{\mathbf{A}}(\vec{a}) = f^{\mathbf{A}}(\vec{a})$ and $t^{\mathbf{B}}(\vec{b}) = f^{\mathbf{B}}(\vec{b})$.

Let $\mathbf{S}$ be the subalgebra of $\mathbf{A} \times \mathbf{B}$ generated by $\{(a_1, b_1), \ldots, (a_n, b_n)\}$. Since $S$ is closed under $f^{\mathbf{A}} \times f^{\mathbf{B}}$ we have that

$$f^{\mathbf{A}} \times f^{\mathbf{B}}((a_1, b_1), \ldots, (a_n, b_n)) = (f^{\mathbf{A}}(\vec{a}), f^{\mathbf{B}}(\vec{b})) \in S.$$

Thus there is a term $t(\vec{x})$ such that

$$(f^{\mathbf{A}}(\vec{a}), f^{\mathbf{B}}(\vec{b})) = t^{\mathbf{A} \times \mathbf{B}}((a_1, b_1), \ldots, (a_n, b_n)) = (t^{\mathbf{A}}(\vec{a}), t^{\mathbf{B}}(\vec{b})).$$

Next we prove by induction in $m$ that:

($\mathrm{I}_m$) If $\mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{K}$ and $\vec{a}_j \in A_j^n$, for $j = 1, \ldots, m$, then there is a term $t(\vec{x})$ satisfying $t^{\mathbf{A}_j}(\vec{a}_j) = f^{\mathbf{A}_j}(\vec{a}_j)$, for $j = 1, \ldots, m$.

By (i) we have that ($\mathrm{I}_m$) holds for $m = 1, 2$. Fix $m \geq 3$, $\mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{K}$ and $\vec{a}_j \in A_j^n$, for $j = 1, \ldots, m$. By inductive hypothesis there are terms $t_1$, $t_2$ and $t_3$ satisfying

$$\begin{aligned} t_1^{\mathbf{A}_j}(\vec{a}_j) &= f^{\mathbf{A}_j}(\vec{a}_j), \text{ for all } j \neq 1,\ 1 \leq j \leq m, \\ t_2^{\mathbf{A}_j}(\vec{a}_j) &= f^{\mathbf{A}_j}(\vec{a}_j), \text{ for all } j \neq 2,\ 1 \leq j \leq m, \\ t_3^{\mathbf{A}_j}(\vec{a}_j) &= f^{\mathbf{A}_j}(\vec{a}_j), \text{ for all } j \neq 3,\ 1 \leq j \leq m. \end{aligned}$$

It is easy to check that $t = M(t_1, t_2, t_3)$ satisfies

$$t^{\mathbf{A}_j}(\vec{a}_j) = f^{\mathbf{A}_j}(\vec{a}_j), \text{ for } j = 1, \ldots, m.$$

We observe that the fact that ($\mathrm{I}_m$) holds for every $m \geq 1$ implies that the hypothesis of Lemma 33 holds and hence $f$ is representable by a term in $\mathcal{K}$. ■

We conclude the section with another term-interpolation result in the spirit of the Baker-Pixley Theorem – in this case, for classes contained in arithmetical varieties having a universal class of finitely subdirectly irreducibles. There are plenty of well-known examples of this kind of varieties, we list a few: f-rings, vector groups, MV-algebras, Heyting algebras, discriminator varieties, etc.

In our proof we use the notion of a *global subdirect product*, which is a special kind of subdirect product, tight enough so that significant information can be obtained from the properties of the factors. We do not provide the definition here but refer the reader to [12].

We write $\mathcal{V}_{FSI}$ to denote the class of finitely subdirectly irreducible members of a variety $\mathcal{V}$.

**Lemma 35** *Let $\mathcal{V}$ be an arithmetical variety of $\mathcal{L}$-algebras and suppose that $\mathcal{V}_{FSI}$ is universal. If $\mathcal{V}_{FSI} \vDash \forall\vec{x}\exists! z\ \varphi(\vec{x}, z)$, where $\varphi \in [\bigwedge \mathrm{At}(\mathcal{L})]$, then there exists an $\mathcal{L}$-term $t(\vec{x})$ such that $\mathcal{V} \vDash \forall\vec{x}\ \varphi(\vec{x}, t(\vec{x}))$.*

**Proof.** By [11] every algebra of $\mathcal{V}$ is isomorphic to a global subdirect product whose factors are finitely subdirectly irreducible. Since global subdirect products preserve $(\forall\exists! \bigwedge p = q)$-sentences (see [15]), we have that $\mathcal{V} \vDash \forall\vec{x}\exists! z\ \varphi(\vec{x}, z)$. Let $\mathbf{F}$ be the algebra of $\mathcal{V}$ freely generated by $x_1, \ldots, x_n$. Since $\mathbf{F} \vDash \exists! z\ \varphi(\vec{x}, z)$, there exists a term $t(\vec{x})$ such that $\mathbf{F} \vDash \varphi(\vec{x}, t(\vec{x}))$. It is easy to check that $\mathcal{V} \vDash \forall\vec{x}\ \varphi(\vec{x}, t(\vec{x}))$. ■

For a class of $\mathcal{L}$-algebras $\mathcal{K}$ let $\mathbb{V}(\mathcal{K})$ denote the *variety generated by* $\mathcal{K}$.

**Theorem 36** *Let $\mathcal{L} \subseteq \mathcal{L}'$ be first order languages without relation symbols and let $f \in \mathcal{L}' - \mathcal{L}$ be an $n$-ary function symbol. Let $\mathcal{K}$ be a class of $\mathcal{L}'$-algebras satisfying:*

*(a) $\mathbb{V}(\mathcal{K}_{\mathcal{L}})$ is arithmetical and $\mathbb{V}(\mathcal{K}_{\mathcal{L}})_{FSI}$ is a universal class.*

*(b) If $\mathbf{A}, \mathbf{B} \in \mathbb{P}_u(\mathcal{K})$ and $\mathbf{S} \leq \mathbf{A}_{\mathcal{L}} \times \mathbf{B}_{\mathcal{L}}$, then $S$ is closed under $f^{\mathbf{A}} \times f^{\mathbf{B}}$.*

*Then $f$ is representable by an $\mathcal{L}$-term in $\mathcal{K}$.*

**Proof.** W.l.o.g. we can suppose that $\mathcal{L}' = \mathcal{L} \cup \{f\}$. As in the proof of Theorem 34 we can see that given $\mathbf{A}, \mathbf{B} \in \mathbb{SP}_u(\mathcal{K})$, $\vec{a} \in A^n$ and $\vec{b} \in B^n$, there is an $\mathcal{L}$-term $t(\vec{x})$ satisfying $t^{\mathbf{A}}(\vec{a}) = f^{\mathbf{A}}(\vec{a})$ and $t^{\mathbf{B}}(\vec{b}) = f^{\mathbf{B}}(\vec{b})$. We establish this property in a wider class.

(i) If $\mathbf{A}, \mathbf{B} \in \mathbb{HSP}_u(\mathcal{K})$, $\vec{a} \in A^n$ and $\vec{b} \in B^n$, then there is an $\mathcal{L}$-term $t(\vec{x})$ such that $t^{\mathbf{A}}(\vec{a}) = f^{\mathbf{A}}(\vec{a})$ and $t^{\mathbf{B}}(\vec{b}) = f^{\mathbf{B}}(\vec{b})$.

Take $\mathbf{A}, \mathbf{B} \in \mathbb{HSP}_u(\mathcal{K})$, and fix $\vec{a} \in A^n$ and $\vec{b} \in B^n$. There are $\mathbf{A}_1 \in \mathbb{SP}_u(\mathcal{K})$, $\mathbf{B}_1 \in \mathbb{SP}_u(\mathcal{K})$ and onto homomorphisms $F : \mathbf{A}_1 \to \mathbf{A}$ and $G : \mathbf{B}_1 \to \mathbf{B}$. Let $\vec{c} \in A_1^n$ and $\vec{d} \in B_1^n$ be such that $F(\vec{c}) = \vec{a}$ and $G(\vec{d}) = \vec{b}$. Let $t(\vec{x})$ be an $\mathcal{L}$-term such that $t^{\mathbf{A}_1}(\vec{c}) = f^{\mathbf{A}_1}(\vec{c})$ and $t^{\mathbf{B}_1}(\vec{d}) = f^{\mathbf{B}_1}(\vec{d})$. Thus, we have that

$$\begin{aligned} t^{\mathbf{A}}(\vec{a}) &= t^{\mathbf{A}}(F(\vec{c})) \\ &= F(t^{\mathbf{A}_1}(\vec{c})) \\ &= F(f^{\mathbf{A}_1}(\vec{c})) \\ &= f^{\mathbf{A}}(F(\vec{c})) \\ &= f^{\mathbf{A}}(\vec{a}), \end{aligned}$$

and similarly, $t^{\mathbf{B}}(\vec{b}) = f^{\mathbf{B}}(\vec{b})$.

Next we prove that

(ii) $\operatorname{Con} \mathbf{A} = \operatorname{Con} \mathbf{A}_{\mathcal{L}}$, for every $\mathbf{A} \in \mathbb{HSP}_u(\mathcal{K})$.

Let $\mathbf{A} \in \mathbb{HSP}_u(\mathcal{K})$ and $\theta \in \operatorname{Con} \mathbf{A}_{\mathcal{L}}$. We show that $\theta$ is compatible with $f$. Suppose $\vec{a}, \vec{b} \in A^n$ are such that $\vec{a}\ \theta\ \vec{b}$. By (i) we have an $\mathcal{L}$-term $t(\vec{x})$ such that $t^{\mathbf{A}}(\vec{a}) = f^{\mathbf{A}}(\vec{a})$ and $t^{\mathbf{A}}(\vec{b}) = f^{\mathbf{A}}(\vec{b})$. Clearly

$$f^{\mathbf{A}}(\vec{a}) = t^{\mathbf{A}}(\vec{a})\ \theta\ t^{\mathbf{A}}(\vec{b}) = f^{\mathbf{A}}(\vec{b}).$$

Now we shall see that

(iii) $\mathbb{HSP}_u(\mathcal{K}_{\mathcal{L}}) \subseteq (\mathbb{HSP}_u(\mathcal{K}))_{\mathcal{L}}$.

It is always the case that $\mathbb{P}_u(\mathcal{K}_{\mathcal{L}}) = \mathbb{P}_u(\mathcal{K})_{\mathcal{L}}$, and (i) implies that $\mathcal{L}$-subreducts of algebras in $\mathbb{P}_u(\mathcal{K})$ are closed under $f$. Thus $\mathbb{SP}_u(\mathcal{K}_{\mathcal{L}}) \subseteq (\mathbb{SP}_u(\mathcal{K}))_{\mathcal{L}}$, and it only remains to see that $\mathbb{H}(\mathbb{SP}_u(\mathcal{K})_{\mathcal{L}}) \subseteq (\mathbb{HSP}_u(\mathcal{K}))_{\mathcal{L}}$, which is immediate by (ii).

By Jónsson's lemma we have that $\mathbb{V}(\mathcal{K}_{\mathcal{L}})_{FSI} \subseteq \mathbb{HSP}_u(\mathcal{K}_{\mathcal{L}})$, and so (iii) produces

(iv) $\mathbb{V}(\mathcal{K}_{\mathcal{L}})_{FSI} \subseteq (\mathbb{HSP}_u(\mathcal{K}))_{\mathcal{L}}$.

Using that $\mathbb{P}_u\mathbb{HSP}_u(\mathcal{K}) \subseteq \mathbb{HSP}_u(\mathcal{K})$ and (i), it is easy to check that $f$ and $\mathbb{HSP}_u(\mathcal{K})$ satisfy the conditions stated in (3) of Theorem 30. Thus we can conclude that

(v) $f = t_1|_{\varphi_1} \cup \ldots \cup t_k|_{\varphi_k}$ in $\mathbb{HSP}_u(\mathcal{K})$, with each $t_i$ an $\mathcal{L}$-term and each $\varphi_i$ in $[\bigwedge \mathrm{At}(\mathcal{L})]$.

Hence,

(vi) the formula $\varphi(\vec{x}, z) = (\varphi_1(\vec{x}) \wedge z = t_1(\vec{x})) \vee \cdots \vee (\varphi_k(\vec{x}) \wedge z = t_k(\vec{x}))$ defines $f$ in $\mathbb{HSP}_u(\mathcal{K})$.

Observe that (v) implies $\mathbb{HSP}_u(\mathcal{K}) \vDash \forall\vec{x}\exists! z\ \varphi(\vec{x}, z)$, and by (iv) we obtain $\mathbb{V}(\mathcal{K}_{\mathcal{L}})_{FSI} \vDash \forall\vec{x}\exists! z\ \varphi(\vec{x}, z)$. Since $\varphi \in [\bigvee \bigwedge \mathrm{At}(\mathcal{L})]$, Proposition 17 says that there is a formula $\psi \in [\bigwedge \mathrm{At}(\mathcal{L})]$ such that $\mathbb{V}(\mathcal{K}_{\mathcal{L}})_{FSI} \vDash \varphi \leftrightarrow \psi$. Thus we have that $\mathbb{V}(\mathcal{K}_{\mathcal{L}})_{FSI} \vDash \forall\vec{x}\exists! z\ \psi(\vec{x}, z)$ and by Lemma 35 there is an $\mathcal{L}$-term $t(\vec{x})$ such that $\mathbb{V}(\mathcal{K}_{\mathcal{L}}) \vDash \forall\vec{x}\ \psi(\vec{x}, t(\vec{x}))$. In particular,

(vii) $\mathbb{V}(\mathcal{K}_{\mathcal{L}})_{FSI} \vDash \forall\vec{x}\ \varphi(\vec{x}, t(\vec{x}))$.

Now, if we take $\mathbf{A} \in \mathbb{V}(\mathcal{K})_{FSI}$, by Jónsson's lemma $\mathbf{A} \in \mathbb{HSP}_u(\mathcal{K})$, and so by (ii) $\mathbf{A}$ and $\mathbf{A}_{\mathcal{L}}$ have the same congruences. Hence $\mathbf{A}_{\mathcal{L}} \in \mathbb{V}(\mathcal{K}_{\mathcal{L}})_{FSI}$, and by (vii) $\mathbf{A} \vDash \forall\vec{x}\ \varphi(\vec{x}, t(\vec{x}))$. Thus,

(viii) $\mathbb{V}(\mathcal{K})_{FSI} \vDash \forall\vec{x}\ \varphi(\vec{x}, t(\vec{x}))$.

Finally, as (vi) says that $\varphi(\vec{x}, z)$ defines $f$ in $\mathbb{HSP}_u(\mathcal{K})$, it follows from (viii) that $\mathbb{V}(\mathcal{K})_{FSI} \vDash \forall\vec{x}\ t(\vec{x}) = f(\vec{x})$. So this identity holds in $\mathbb{V}(\mathcal{K})$ and the theorem is proved. ■

**Corollary 37** *Let $\mathcal{V}$ be an arithmetical variety such that $\mathcal{V}_{FSI}$ is universal. Let $\mathcal{K} \subseteq \mathcal{V}$ be a first order class and suppose that $\psi(\vec{x}, z)$ is a first order formula (in the language of $\mathcal{V}$) which defines on each algebra $\mathbf{A}$ of $\mathcal{K}$ a function $f^{\mathbf{A}} : A^n \to A$. Assume that for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ every subalgebra $\mathbf{S} \leq \mathbf{A} \times \mathbf{B}$ is closed under $f^{\mathbf{A}} \times f^{\mathbf{B}}$. Then $f$ is representable by a term in $\mathcal{K}$.*

We believe it likely that the Baker-Pixley Theorem holds in scenarios other than the two considered here (locally finite and arithmetical). A question we were unable to answer is the following.

Let $\mathcal{K}$ be a first order axiomatizable class of $(\mathcal{L} \cup \{f\})$-algebras with a majority $\mathcal{L}$-term and suppose that for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ every subalgebra of $\mathbf{A}_{\mathcal{L}} \times \mathbf{B}_{\mathcal{L}}$ is closed under $f^{\mathbf{A} \times \mathbf{B}}$. Is $f$ representable by an $\mathcal{L}$-term in $\mathcal{K}$?

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